ELSEVIER Journal of Geometry and Physics 16 (1995) 275–300

JOURNAL OF GEOMETRY AND PHYSICS

Infinite dimensional geometry and quantum field theory of strings. III. Infinite dimensional W-geometry of a second quantized free string

D.V. Juriev [1]

Laboratoire de Physique Théorique de l'École Normale Supérieure, 24 rue Lhomond, 75231 Paris Cedex 05, France

e-mail: `juriev@physique.ens.fr` [2]

Received 11 January 1994; revised 30 June 1994

Abstract

The present paper is devoted to various objects of the infinite dimensional W-geometry of a second quantized free string. Our purpose is to include the W-symmetries into the general infinite dimensional geometrical picture related to the quantum field theory of strings, which was described in part I of this series of papers. It is done by replacing the Lie algebra of all infinitesimal reparametrizations of a string world-sheet by the Lie quasi(pseudo)algebra of classical W-transformations (Gervais–Matsuo quasi(pseudo)algebra) as well as the Virasoro algebra by the central extended enlarged Gervais–Matsuo quasi(pseudo)algebra. A way to obtain W-algebras from the classical W-transformations (i.e. Gervais–Matsuo Lie quasi(pseudo)algebra) is proposed. A relationship between Gervais–Matsuo differential W-geometry and the Batalin–Weinstein–Karasev–Maslov approach to the geometry of nonlinear Poisson brackets as well as L.V. Sabinin's program of "nonlinear geometric algebra" is mentioned.

Keywords: Infinite dimensional geometry; String field theory; W-symmetries and W-geometry; Lie quasi(pseudo)algebras

1991 MSC: 58 B 25, 17 B 65, 81 T 30, 17 D 99

[1] On leave from Mathematical Division of Research Institute for System Studies, Moscow, Russia (email: `juriev@systud.msk.su`).

[2] Unité Propre du Centre National de la Recherche Scientifique associée à l'École Normale Supérieure et à l'Université de Paris-Sud.

SSDI 0393-0440(94)00030-1

0. Introduction

This is the third part of the text devoted to an infinite dimensional geometry of a quantum field theory of strings (for previous ones see Refs. [1,2]). This part concerns an infinite dimensional geometry, which appears in the theory of second quantized free strings, as well as the first one. An origin of this article lies in some very crucial detail of the theory, which is known under the title of W-symmetries. The algebraic aspects of W-symmetries, somewhat cumbersome and mysterious at the beginning [3], were recently clarified by B.A. Khesin, V.Yu. Ovsienko, A.O. Radul et al. [4–10]. Moreover, a simple geometric interpretation of them was given by J.-L. Gervais and Y. Matsuo [11]. It seems that the cruciality of W-symmetries to the theory of second quantized free strings lies in the fact that W-"reparametrizations" of strings, which are considered, may be not obligatory global in the loop space (i.e. the same for all strings independently of their position in a target space) but, in some sense, local, i.e. depend in some natural way on an embedding of the string into the target space. One may say that such transformations are related not only to the intrinsic but also extrinsic geometry of a string. A class of the most interesting transformations of this kind, which are related to a certain complex analog of a classical Frenet theory of curve invariants [12], was described by J.-L. Gervais and Y. Matsuo as classical W-transformations. Such transformations of both internal and external degrees of freedom of a string may be considered hidden ones with respect to the standard conformal symmetries. So a natural question arises: how the process of the geometric quantization of a string changes on account of these hidden symmetries. From the mathematical point of view it may be considered to be a problem of an "induction" of the quantization process directed by a considered symmetry algebra to one directed by its extension. So the solution of this problem (reproduced from a typical pattern for this partial infinite dimensional case of the quantum field theory of free strings, which we consider) should produce a lot of new geometric material, completing a general picture, which was drawn in [1], with new intriguing details and nuances.

This opinion is also confirmed by the fact that W_∞-symmetries are very deeply related to the group of area-preserving diffeomorphisms of a torus. This group, the corresponding Lie algebra, its deformations and central extensions, being the main objects of a certain development of string theory and membrane theory, produce the next scope of infinite dimensional objects after that of the Virasoro algebra. So we may consider the geometric material, which will be discussed in this paper, as a possible introduction to the transition from the careful consideration of the infinite dimensional geometry of objects from the "Virasoro family", which was begun by A.A. Kirillov and the author in [13] and summarized partially in [14,1,2] (see also references therein) to a systematic treatment of geometric material, concerning objects related to the group of symplectomorphisms of a torus.

Such a treatment is supposed to be presented in one of the forthcoming publications.

In the aspect of the applications of our geometric material we follow the general ideology of A.Yu. Morozov, formulated in [15]. He proposed to handle string theory not as a partial physical theory describing some narrow class of real phenomena but as one of the universal theories, whose final and, it seems, rather wide area of applications is not completely understood now. It is possible that some new applications of certain aspects of the string theory (and of an infinite dimensional geometry related to it) will be found during an interdisciplinary "toy-program" of investigating the peculiarities of human vision (in particular, of color perception) in artificial interactive computer graphic systems, formulated by the author in [16], the results of which may be important for the understanding of various processes in natural interactive (sensorial and visual) systems. Hence, our presentation is maximally mathematical and avoids many technical aspects, which are crucial for some applications; instead of that we prefer to specially point out a general mathematical meaning of our constructions.

1. Symplectomorphisms (area-preserving transformations) of a torus, differential operators on a circle and W-algebras

1.1. Groups and algebras

In this subsection we discuss

- the groups $\mathrm{Symp}(\mathbb{T}^2)$, $\mathrm{Symp}^e(\mathbb{T}^2)$ and $\mathrm{Symp}_0(\mathbb{T}^2)$ of symplectomorphisms of a torus $\mathbb{T}^2$ and their Lie algebras $\mathrm{Ham}(\mathbb{T}^2)$ and $\mathrm{Ham}_0(\mathbb{T}^2)$ of a Hamiltonian and strictly Hamiltonian vector field on a torus;
- the Poisson algebras $\mathcal{F}^{\mathbb{C}}(\mathbb{T}^2)$, $\mathcal{F}^{\mathbb{C}}(T^*(S^1))$ and $\mathcal{F}^{\mathbb{C}}(\mathbb{R}^2)$ of functions on torus, cylinder and plane, their deformations and central extensions.

Let $(\mathbb{T}^2,\omega)$ be a two-dimensional torus with a fixed 2-form (a volume form) ω on it. Let $\mathrm{Symp}(\mathbb{T}^2,\omega)$ be *a group of symplectomorphisms of a torus* $(\mathbb{T}^2,\omega)$, i.e. all diffeomorphisms ζ of $\mathbb{T}^2$ such that $\zeta_*\omega = \omega$ [17]. So symplectomorphisms of a torus are just *its area-preserving transformations.*

It may be easily shown that all groups $\mathrm{Symp}(\mathbb{T}^2,\omega)$ are isomorphic. Indeed, let us identify a torus $(\mathbb{T}^2,\omega)$ with the quotient $\mathbb{R}^2/\Gamma$, where the plane $\mathbb{R}^2$ with a fixed coordinate system (p,q) possesses a canonical 2-form $dp\wedge dq$, Γ is a certain free lattice on $\mathbb{R}^2$, i.e. a set $\mathbb{Z}\boldsymbol{a}+\mathbb{Z}\boldsymbol{b}$, where $\boldsymbol{a}$ and $\boldsymbol{b}$ are two independent vectors on the plane. The symplectomorphisms of a torus may be identified with the classes $\mathrm{mod}\,\Gamma$ of functions $\boldsymbol{f}:\mathbb{R}^2\mapsto\mathbb{R}^2$, which obey some conditions. The expression $\mathrm{mod}\,\Gamma$ means that two functions $\boldsymbol{f}_1$ and $\boldsymbol{f}_2$ are called equivalent iff $\boldsymbol{f}_1(\boldsymbol{x})-\boldsymbol{f}_2(\boldsymbol{x})\in\Gamma$ for all $\boldsymbol{x}$ from $\mathbb{R}^2$. It should be mentioned that in the coordinate form a function $\mathbf{f}$ is represented as a pair (f_p,f_q) of functions $f_p=f_p(p,q)$ and $f_q=f_q(p,q)$. The conditions, that a function $\boldsymbol{f}$ should satisfy, are the following ones:

(i) The periodicity condition: $\boldsymbol{f}(\boldsymbol{x}+m\boldsymbol{a}+n\boldsymbol{b})=\boldsymbol{f}(\boldsymbol{x})+m\tilde{\boldsymbol{a}}+n\tilde{\boldsymbol{b}}$ for all $\boldsymbol{x}$ from $\mathbb{R}^2$, where $\tilde{\boldsymbol{a}}$ and $\tilde{\boldsymbol{b}}$ is a pair of generators of Γ, i.e. $\tilde{\boldsymbol{a}}=A(\boldsymbol{a})$, $\tilde{\boldsymbol{b}}=A(\boldsymbol{b})$, for some transformation A from the group $\mathrm{Aut}(\Gamma)$, isomorphic to $\mathrm{SL}(2,\mathbb{Z})$;

(ii) The normalization of Jacobian condition: $\boldsymbol{Df}/\boldsymbol{Dx}=\det\partial(f_p,f_q)/\partial(p,q)=1.$

The last determinant (Jacobian) is invariant under all linear transformations of a plane $\mathbb{R}^2$. Since for any two free lattices Γ_1 and Γ_2, there exists a linear transformation γ of $\mathbb{R}^2$ such that $\Gamma_2 = \gamma(\Gamma_1)$, the groups of symplectomorphisms of $\mathbb{R}^2/\Gamma_1$ and $\mathbb{R}^2/\Gamma_2$ may be identified. So instead of the notation $\mathrm{Symp}(\mathbb{T}^2, \omega)$ for a group of symplectomorphisms of a torus we shall use the simpler notation $\mathrm{Symp}(\mathbb{T}^2)$.

It should be mentioned that the group $\mathrm{Symp}(\mathbb{T}^2)$ is not connected. Let us denote the component of the identity of this group by $\mathrm{Symp}^e(\mathbb{T}^2)$. Of course, $\mathrm{Symp}^e(\mathbb{T}^2)$ is a normal subgroup in $\mathrm{Symp}(\mathbb{T}^2)$ and therefore we have the following exact sequence:

$$0 \longrightarrow \mathrm{Symp}^e(\mathbb{T}^2) \longrightarrow \mathrm{Symp}(\mathbb{T}^2) \longrightarrow \mathrm{Symp}(\mathbb{T}^2)/\,\mathrm{Symp}^e(\mathbb{T}^2) \longrightarrow 0.$$

The quotient $\mathrm{Symp}(\mathbb{T}^2)/\,\mathrm{Symp}^e(\mathbb{T}^2)$ is isomorphic to *the modular group* $\mathrm{SL}(2,\mathbb{Z})$, so that the functions f that correspond to the elements of $\mathrm{Symp}^e(\mathbb{T}^2)$ satisfy the periodicity condition of the form $\boldsymbol{f}(\boldsymbol{x} + m\boldsymbol{a} + n\boldsymbol{b}) = \boldsymbol{f}(\boldsymbol{x}) + m\boldsymbol{a} + n\boldsymbol{b}$.

The group $\mathrm{Symp}^e(\mathbb{T}^2)$ has a normal subgroup $\mathrm{Symp}_0(\mathbb{T}^2)$ of symplectomorphisms ζ such that

$$\int\limits_{-1/2}^{1/2}\int\limits_{-1/2}^{1/2} \boldsymbol{f}(\boldsymbol{x} + t_1\boldsymbol{a} + t_2\boldsymbol{b})\, dt_1 dt_2 = \boldsymbol{x}\,.$$

The quotient $\mathrm{Symp}^e(\mathbb{T}^2)/\,\mathrm{Symp}_0(\mathbb{T}^2)$ is isomorphic to 2-dimensional compact abelian group $\mathbb{T}^2$, so that an exact sequence

$$0 \longrightarrow \mathrm{Symp}_0(\mathbb{T}^2) \longrightarrow \mathrm{Symp}^e(\mathbb{T}^2) \longrightarrow \mathbb{T}^2 \longrightarrow 0$$

exists. This exact sequence may be split, the splitting map $\mathbb{T}^2 \to \mathrm{Symp}^e(\mathbb{T}^2)$ realizing a group $\mathbb{T}^2$ as a group of movements of torus $\mathbb{T}^2$.

The Lie algebra of the group $\mathrm{Symp}^e(\mathbb{T}^2)$ is *an algebra of Hamiltonian vector fields on a torus*, i.e. the fields ξ such that $\mathcal{L}_\xi\omega = 0$. It should be mentioned that after an identification of a torus $\mathbb{T}^2$ with the quotient $\mathbb{R}^2/\Gamma$ Hamiltonian fields ξ on a torus may be also characterized as divergence free ones, i.e. vector fields ξ such that $\operatorname{div}\xi = 0$. It follows from Cartan Theorem that ξ is Hamiltonian iff $d\alpha_\xi = 0$, where $\alpha_\xi = \iota_\xi\omega$. So the algebra $\mathrm{Ham}(\mathbb{T}^2)$ of Hamiltonian vector fields on a torus may be identified with an algebra of closed 1-forms on a torus with respect to the bracket $\{\alpha, \beta\} = d(\alpha\wedge\beta)/\omega$.

The Lie algebra of the group $\mathrm{Symp}_0(\mathbb{T}^2)$ is a subalgebra of $\mathrm{Symp}^e(\mathbb{T}^2)$, *the algebra of strictly Hamiltonian vector fields*, i.e. the Hamiltonian vector fields ξ such that the corresponding 1-form α_ξ is exact. The Lie algebra $\mathrm{Ham}_0(\mathbb{T}^2)$ of exact Hamiltonian vector fields on a torus is an ideal in the algebra $\mathrm{Ham}(\mathbb{T}^2)$ so that the following exact sequence

$$0 \longrightarrow \mathrm{Ham}_0(\mathbb{T}^2) \longrightarrow \mathrm{Ham}(\mathbb{T}^2) \longrightarrow \mathbb{R}^2 \longrightarrow 0$$

exists. The quotient $\mathbb{R}^2 = \mathrm{Ham}(\mathbb{T}^2)/\,\mathrm{Ham}_0(\mathbb{T}^2)$ may be interpreted as the cohomology group $H^1(\mathbb{T}^2, \mathbb{R})$ if we consider it the quotient of a space of closed 1-forms by a subspace of exact ones. This exact sequence may be split. The splitting map $\mathbb{R}^2 \longmapsto \mathrm{Ham}(\mathbb{T}^2)$ realizes the vector space $\mathbb{R}^2$ as a certain parallelization of a torus.

It should be mentioned that if 1-form α_ξ is exact then there exists a function F_ξ on a torus such that $\alpha_\xi = dF_\xi$. The bracket in the space of exact Hamiltonian vector fields induces a bracket in the space $\mathcal{F}(\mathbb{T}^2)$ of functions (0-forms) on a torus. The bracket in the space $\mathcal{F}(\mathbb{T}^2)$ is of the form $\{F, G\} = (dF \wedge dG)/\omega$. The center of the Lie algebra $\mathcal{F}(\mathbb{T}^2)$ consists of constant functions, so it may be identified with 1-dimensional space $\mathbb{R}$. The Lie algebra of exact Hamiltonian vector fields on a torus is isomorphic to a quotient $\mathcal{F}(\mathbb{T}^2)/\mathbb{R}$. The isomorphism is realized by the exterior derivative d, which maps the above quotient onto the Lie algebra of exact 1-forms, which is isomorphic to $\mathrm{Ham}_0(\mathbb{T}^2)$. The mapping from the algebra $\mathcal{F}(\mathbb{T}^2)$ to the algebra $\mathrm{Ham}_0(\mathbb{T}^2)$ is denoted by sgrad and is called "*skew-gradient*", so that $\xi = \mathrm{sgrad}(F_\xi)$ for all exact Hamiltonian vector fields on a torus. It should be mentioned that the exact sequence

$$0 \longmapsto \mathbb{R} \longmapsto \mathcal{F}(\mathbb{T}^2) \longmapsto \mathrm{Ham}_0(\mathbb{T}^2) \longmapsto 0$$

may be split. The splitting map identifies the algebra $\mathrm{Ham}_0(\mathbb{T}^2)$ with the subspace $\mathcal{F}_0(\mathbb{T}^2)$ of functions of zero mean value.

The bracket $\{\cdot,\cdot\}$ on the space $\mathcal{F}(\mathbb{T}^2)$ of functions on a torus defines Poisson algebra structure on it. This algebra is called *the Poisson algebra of functions on a torus*. It means that $\{F, GH\} = \{F, G\}H + G\{F, H\}$ for all functions F, G, H from $\mathcal{F}(\mathbb{T}^2)$. In other words the bracket $\{\cdot,\cdot\}$ is a derivation of the ordinary commutative multiplication of functions.

Let us represent the torus $\mathbb{T}^2$ as a quotient $\mathbb{R}^2/\Gamma$, where $\Gamma = \{(mM, nN) \in \mathbb{R}^2, m, n \in \mathbb{Z}; M, N$ are fixed real positive numbers$\}$. Then there is a canonical basis in $\mathcal{F}^{\mathbb{C}}(\mathbb{T}^2)$, the complexification of the algebra $\mathcal{F}(\mathbb{T}^2)$: $e_{m,n} = \mathcal{D}(M, N)\exp(imp/M)\exp(inq/N)$, where it is convenient to put a normalization constant $\mathcal{D}(M, N)$ equal MN. The Poisson brackets $\{\cdot,\cdot\}$ in $\mathcal{F}(\mathbb{T}^2)$ or $\mathcal{F}^{\mathbb{C}}(\mathbb{T}^2)$ have the form

$$\{F, G\} = \partial F/\partial p\, \partial G/\partial q - \partial G/\partial p\, \partial F/\partial q$$

so that [17]

$$\{e_{m,n}, e_{m',n'}\} = (mn' - nm')e_{m+m',n+n'}.$$

The subalgebra $\mathcal{F}_0^{\mathbb{C}}(\mathbb{T}^2)$ of $\mathcal{F}^{\mathbb{C}}(\mathbb{T}^2)$ (the complexification of $\mathcal{F}_0(\mathbb{T}^2)$) is realized as one spanned by generators $e_{m,n}$, $(m, n) \neq (0,0)$.

Let us call the algebra spanned formally by $e_{m,n}$, the *Floratos–Iliopoulos algebra* (see [18], where it was introduced). It should be mentioned that generators L_k, $k \in \mathbb{Z}$ of the algebra $\mathbb{C}\mathrm{Vect}(\mathbb{S}^1)$ of vector fields on a circle $\mathbb{S}^1$ satisfying the commutation relations $[L_i, L_j] = (i - j)L_{i+j}$ may be expressed formally via $e_{m,n}$ as follows [18]:

$$L_k = \sum_{m \in \mathbb{Z}} \mathcal{G}_k(m) e_{m,k},$$

where $\mathcal{G}_k(m) = (-1)^m/m$ if $m \neq 0$ and 0 otherwise. Of course, in order for the commutation relations of the generators L_k to hold some (rather natural) regularization must be used. So the Lie algebra $\mathbb{C}\mathrm{Vect}(\mathbb{S}^1)$ (more precisely, the so called *Witt algebra*, which is spanned by the generators L_k of $\mathbb{C}\mathrm{Vect}(\mathbb{S}^1)$, but we shall not distinguish between them below) is realized as a certain subalgebra of the Floratos–Iliopoulos algebra.

With respect to the action of the algebra $\mathbb{C}\mathrm{Vect}(S^1)$, the Floratos–Iliopoulos algebra may be decomposed into a sum of tensor modules of positive integer weights, the corresponding tensor operators will be denoted by $w_n^{(i)}$ $(i \geq 2)$. Note that $L_n = w_n^{(2)}$ and the commutation relations between the generators $w_n^{(i)}$ have the form

$$[w_n^{(i)}, w_m^{(j)}] = ((j-1)n - (i-1)m) w_{n+m}^{(i+j-2)},$$

the $w_n^{(i)}$ themselves being defined as

$$w_k^{(i)} = \sum_{k \in \mathbb{Z}} \mathcal{G}_k^{(i)}(m) e_{k,m},$$

$\mathcal{G}_k^{(i)}(m) = P_i(1/m)$, where the polynomials $P_i(x)$ of degree $i-1$ are connected by the easily calculated recurrence formulas.

The algebra $\mathrm{Ham}_0(\mathbb{T}^2)$ of exact Hamiltonian vector fields on a torus admits a central extension $\widehat{\mathrm{Ham}}_0(\mathbb{T}^2)$, which may be defined as follows: $[\xi,\eta]_\partial = [\xi,\eta] + \int_{\mathbb{T}^2} [\partial, \xi] \eta \, dp \, dq$, where ∂ is a certain Hamiltonian but not strictly Hamiltonian vector field on the torus. One may choose $\partial = a\partial_p + b\partial_q$ and this is a general setting up to trivial central extensions. So the universal central extension of $\mathrm{Ham}_0(\mathbb{T}^2)$ is defined by the following exact sequence:

$$0 \longrightarrow H_1(\mathbb{T}^2, \mathbb{R}) \longrightarrow \widehat{\mathrm{Ham}}_0(\mathbb{T}^2) \longrightarrow \mathrm{Ham}_0(\mathbb{T}^2) \longrightarrow 0,$$

where $H_1(\mathbb{T}^2, \mathbb{R})$ is a 2-dimensional homology group of the torus with real coefficients.

The central extension $\widehat{\mathrm{Ham}}_0(\mathbb{T}^2)$ of the Lie algebra $\mathrm{Ham}_0(\mathbb{T}^2)$ induces a central extension $\widehat{\mathcal{F}}^{\mathbb{C}}(\mathbb{T}^2)$ of the algebra $\mathcal{F}^{\mathbb{C}}(\mathbb{T}^2)$, defined as $\{F,G\}_\partial = \{F,G\} + \int_{\mathbb{T}^2} \mathcal{L}_\partial F \cdot G \, dp \, dq$, so that the following exact sequence:

$$0 \longrightarrow \mathbb{C}^2 \longrightarrow \widehat{\mathcal{F}}^{\mathbb{C}}(\mathbb{T}^2) \longrightarrow \mathcal{F}^{\mathbb{C}}(\mathbb{T}^2) \longrightarrow 0$$

exists. This central extension is of the following form in the basis $e_{m,n}$:

$$[e_{m,n}, e_{m',n'}] = (mn' - nm') e_{m+m',n+n'} + (am + bn)\delta(m+m')\delta(n+n')\mathbf{1}.$$

The extended Floratos–Iliopoulos algebra admits an embedding of the Virasoro algebra $\mathbb{C}\mathrm{vir}$ with a non-trivial central charge (one of the possible embeddings was described in Ref. [18]).

The Poisson algebra $\mathcal{F}^{\mathbb{C}}(\mathbb{T}^2)$ may be deformed into an associative algebra [19]. In the canonical basis such a deformation has the form

$$e_{m,n} \cdot e_{m',n'} = \exp(i\lambda(mn' - nm')) e_{m+m',n+n'}.$$

The corresponding commutator algebra has the form (after a renormalization of generators)

$$[e_{m,n}, e_{m',n'}] = \frac{1}{\lambda} \sin(\lambda(nm' - mn')) e_{m+m',n+n'}.$$

So there is defined a certain deformation of Floratos–Iliopoulos algebra, which is called (centerless) *sine-algebra (or Fairlie–Fletcher–Zachos algebra)* [20]. The corresponding deformation of $\mathrm{Ham}_0(\mathbb{T}^2)$ is called *the Lie algebra of exact Hamiltonian vector fields on a quantum torus* and is denoted by $\mathrm{Ham}_0(\mathbb{T}^2_q)\,(q=e^{i\lambda})$.

The (centerless) sine algebra admits a central extension of the form

$$[e_{m,n}, e_{m',n'}] = \frac{1}{\lambda}\sin(\lambda(nm'-mn'))e_{m+m',n+n'} + (am+bn)\delta(m+m')\delta(n+n')\mathbf{1}.$$

The sine-algebra with a center also admits an embedding of the Virasoro algebra $\mathbb{C}\mathrm{vir}$:

$$L_k = \sum_{m\in\mathbb{Z}} \widetilde{\mathcal{G}}_k(m)e_{k,m},$$

where the $\widetilde{\mathcal{G}}_k(m)$ are defined by formulas, analogous to the ones of Ref. [18].

With respect to the action of the algebra $\mathbb{C}\mathrm{vir}$, the sine-algebra may also be decomposed into the sum of tensor modules of positive integer weights. The corresponding tensor operators will be denoted by $\tilde{w}_n^{(i)}$ $(i\geq 2)$. Their commutation relations are of the form (for the centerless Sine-algebra):

$$[\omega_n^{(k)}, \omega_m^{(l)}] = \sum_{p\geq 0} A_{kl}^p \omega_{m+n}^{(k+l-2p-1)},$$

where

$$A_{kl}^p = \sum_{i+j=2p+1} (-1)^i C_{k-1}^i C_{l-1}^j,$$

$$C_n^m = \frac{n!}{m!\,(n-m)!} \quad \text{if } m\leq n, \qquad C_n^m = 0 \quad \text{otherwise}.$$

Now let us mention that all our constructions admit certain limits when one or both parameters N and M tend to infinity. That means that the torus becomes a cylinder or a plane. In the case $M\to\infty$ the Poisson algebra $\mathcal{F}^{\mathbb{C}}(\mathbb{T}^2)$ inverts into *the Poisson algebra $\mathcal{F}^{\mathbb{C}}(T^*\mathbb{S}^1)$ of functions on a cylinder.* There is natural basis in $\mathcal{F}^{\mathbb{C}}(T^*\mathbb{S}^1)$: $e_n^m = -i\exp(inq/N)p^m$, in which the Poisson brackets have the form:

$$\{e_n^m, e_{n'}^{m'}\} = (mn'-nm')e_{n+n'}^{m+m'-1}.$$

The embedding of the algebra $\mathbb{C}\mathrm{Vect}(\mathbb{S}^1)$ into $\mathcal{F}^{\mathbb{C}}(T^*\mathbb{S}^1)$ is natural: $L_k\mapsto e_k^1$. Such an embedding admits in fact a one-parametric deformation $L_k\mapsto e_k^1+(k+1)\lambda e_k^0$ where λ is a parameter. The central extension $\widehat{\mathcal{F}}^{\mathbb{C}}(T^*\mathbb{S}^1)$ of $\mathcal{F}^{\mathbb{C}}(T^*\mathbb{S}^1)$ is of the form

$$\{e_n^m, e_{n'}^{m'}\} = (mn'-nm')e_{n+n'}^{m+m'-1} + \tfrac{1}{12}(n^3-n)\delta(n+n')\delta_{m,1}\delta_{m',1}\mathbf{1}.$$

The embeddings of $\mathbb{C}\mathrm{Vect}(\mathbb{S}^1)$ into $\mathcal{F}^{\mathbb{C}}(T^*\mathbb{S}^1)$ extends to embeddings of the Virasoro algebra $\mathbb{C}\mathrm{vir}$ into $\widehat{\mathcal{F}}^{\mathbb{C}}(T^*\mathbb{S}^1)$.

The Poisson algebra $\mathcal{F}^{\mathbb{C}}(T^*\mathbb{S}^1)$ is deformed into *the algebra* $\mathrm{DOP}^{\mathbb{C}}(\mathbb{S}^1)$ *of differential operators on a circle.* The corresponding commutator algebra is *a Lie algebra*

$\mathrm{DOP}^{\mathbb{C}}_{[\cdot,\cdot]}(\mathbb{S}^1)$ *of differential operators on a circle.* The embeddings of $\mathbb{C}\mathrm{Vect}(\mathbb{S}^1)$ in $\mathcal{F}^{\mathbb{C}}(T^*\mathbb{S}^1)$ induce embeddings of this algebra into $\mathrm{DOP}^{\mathbb{C}}_{[\cdot,\cdot]}(\mathbb{S}^1)$; these embeddings are the same if one considers a subalgebra $\mathrm{DOP}^{\mathbb{C}}_{[\cdot,\cdot]}(\mathbb{S}^1)_+$ of $\mathrm{DOP}^{\mathbb{C}}_{[\cdot,\cdot]}(\mathbb{S}^1)$, the commutator algebra of the algebra $\mathrm{DOP}^{\mathbb{C}}(\mathbb{S}^1)_+$ of differential operators on the circle $\mathbb{S}^1$ without free terms. Both algebras $\mathrm{DOP}^{\mathbb{C}}_{[\cdot,\cdot]}(\mathbb{S}^1)$ and $\mathrm{DOP}^{\mathbb{C}}_{[\cdot,\cdot]}(\mathbb{S}^1)_+$ admit central extensions $\widehat{\mathrm{DOP}}^{\mathbb{C}}_{[\cdot,\cdot]}(\mathbb{S}^1)$ and $\widehat{\mathrm{DOP}}^{\mathbb{C}}_{[\cdot,\cdot]}(\mathbb{S}^1)_+$, which are defined by *the Kac-Peterson cocycle*

$$c(f(q)D^m, g(q)D^n) = \frac{m!\,n!}{(m+n+1)!}\int_{\mathbb{S}^1} f^{(n)}(q)g^{(m+1)}(q)\,dq,$$

where $D = \partial/\partial q$, $f^{(n)}(q) = \partial^n f(q)/\partial q^n$ [21] (see also Ref. [6]). The embeddings of $\mathbb{C}\mathrm{Vect}(\mathbb{S}^1)$ in $\mathrm{DOP}^{\mathbb{C}}_{[\cdot,\cdot]}(\mathbb{S}^1)$ (resp. the embedding in $\mathrm{DOP}^{\mathbb{C}}_{[\cdot,\cdot]}(\mathbb{S}^1)_+$) are extended to embeddings of the Virasoro algebra $\mathbb{C}\mathrm{vir}$ in $\widehat{\mathrm{DOP}}^{\mathbb{C}}_{[\cdot,\cdot]}(\mathbb{S}^1)$ (resp. an embedding in $\widehat{\mathrm{DOP}}^{\mathbb{C}}_{[\cdot,\cdot]}(\mathbb{S}^1)_+$).

To make the construction of central extensions $\widehat{\mathrm{DOP}}^{\mathbb{C}}_{[\cdot,\cdot]}(\mathbb{S}^1)$ and $\widehat{\mathrm{DOP}}^{\mathbb{C}}_{[\cdot,\cdot]}(\mathbb{S}^1)_+$ of $\mathrm{DOP}^{\mathbb{C}}_{[\cdot,\cdot]}(\mathbb{S}^1)$ and $\mathrm{DOP}^{\mathbb{C}}_{[\cdot,\cdot]}(\mathbb{S}^1)_+$ more clear one should transit to *the algebra* $\mathrm{PDOP}^{\mathbb{C}}(\mathbb{S}^1)$ *of all pseudodifferential operators on a circle* $\mathbb{S}^1$ [5-8]. The commutator algebra $\mathrm{PDOP}^{\mathbb{C}}_{[\cdot,\cdot]}(\mathbb{S}^1)$ of this algebra, *the Lie algebra of pseudodifferential operators on a circle*, admits a central extension, which may be defined as [5]

$$[A,B]_c = [A,B] + c\cdot\mathrm{Tr}([A,\log D],B]),$$

where

$$\mathrm{Tr}(A) = \int \mathrm{Res}(A)\,dq, \quad \mathrm{Res}(A) = a_{-1}(q) \text{ if } A = \sum_{k\in\mathbb{Z}} a_k(q)D^k.$$

The cocycle $c(A,B) = \mathrm{Tr}([A,\log D],B)$ is the so-called *Kravchenko-Khesin cocycle.* The extension by the Kravchenko-Khesin cocycle being restricted to subalgebras $\mathrm{DOP}^{\mathbb{C}}_{[\cdot,\cdot]}(\mathbb{S}^1)$ and $\mathrm{DOP}^{\mathbb{C}}_{[\cdot,\cdot]}(\mathbb{S}^1)_+$ defines extensions described above.

In the case $M,N\to\infty$ the Poisson algebra $\mathcal{F}^{\mathbb{C}}(\mathbb{T}^2)$ inverts into the Poisson algebra $\mathcal{F}^{\mathbb{C}}(\mathbb{R}^2)$ of functions on a plane $\mathbb{R}^2$. There is a natural polynomial basis in $\mathcal{F}^{\mathbb{C}}(\mathbb{R}^2)$: $e^{mn} = p^m q^n$, in which the Poisson brackets have the form

$$\{e^{mn}, e^{m'n'}\} = (mn' - nm')e^{m+m'-1,n+n'-1}.$$

The Poisson algebra $\mathcal{F}^{\mathbb{C}}(\mathbb{R}^2)$ may be deformed into the Weyl algebra $W_{\mathbb{C}}(\mathbb{R}^2)$. Polynomial elements of this algebra may be represented by differential operators on a line $\mathbb{R}^1$, so there exists a differential operator $\mathcal{D}_P$ corresponding to the polynomial P from $\mathcal{F}^{\mathbb{C}}(\mathbb{R}^2)$, where P is a Weyl (symmetric) symbol of $\mathcal{D}_P$. The commutator algebra of the Weyl algebra $W_{\mathbb{C}}(\mathbb{R}^2)$ is the so-called Moyal algebra $\mathrm{Moy}_{\mathbb{C}}(\mathbb{R}^2)$. The commutator in the Moyal algebra is of the form [19,22]

$$[F,G] = \sum_{p\in\mathbb{Z}}(-1)^p \sum_{i+j=2p+1} \frac{(-1)^i}{i!\,j!}(\partial_p^i\partial_q^j F)(\partial_p^j\partial^i G)$$

or, in the basis e^{mn},

$$[e^{mn}, e^{m'n'}] = \sum_{p\geq 0}(-1)^p A^{2p+1}_{mm'nn'} e^{m+m'-2p-1,n+n'-2p-1},$$

where

$$A^p_{mm'nn'} = \sum_{i+j=p}(-1)^i i!\, j!\, C^i_n C^j_n C^j_{m'} C^i_{n'},$$

$$C^m_n = \frac{n!}{m!\,(n-m)!} \quad \text{if } m \leq n, \qquad C^m_n = 0 \quad \text{otherwise}.$$

1.2. Algebras

In this subsection we discuss

- the Poisson algebra $\mathcal{F}^{\mathbb{C}}(T^*\mathbb{S}^1)$, its central extension $\widehat{\mathcal{F}}^{\mathbb{C}}(T^*\mathbb{S}^1)$ and w_∞-algebra (Bakas algebra);
- the Lie algebra $\mathrm{DOP}^{\mathbb{C}}_{[\cdot,\cdot]}(\mathbb{S}^1)$, its central extension $\widehat{\mathrm{DOP}}^{\mathbb{C}}_{[\cdot,\cdot]}(\mathbb{S}^1)$ and W_∞-algebras (Pope–Shen–Romans algebras);
- the Gelfand–Dickey algebras $\mathrm{GD}^{\mathbb{C}}_n$ and Radul mapping $\mathcal{F}(\mathrm{DOP}^{\mathbb{C}}_{[\cdot,\cdot]}(\mathbb{S}^1)^*) \mapsto \mathrm{GD}^{\mathbb{C}}_n$;
- the Radul bundle $\mathrm{Rad}^{\mathbb{C}}(\mathcal{M}_n)$ and the Lie algebra of its sections $\mathrm{Rad}_{\mathbb{C}}$ (Radul algebra);
- the wedge subalgebra of the Pope–Sheen–Romans W_∞-algebra;
- the associative model algebras $\mathrm{Md}^{(\lambda)}(\mathrm{sl}(2,\mathbb{C}))$ for the Lie algebra $\mathrm{sl}(2,\mathbb{C})$ and their commutator algebras $\mathrm{Md}^{(\lambda)}_{[\cdot,\cdot]}(\mathrm{sl}(2,\mathbb{C}))$— the model Lie algebras (or Feigin algebras) for $\mathrm{sl}(2,\mathbb{C})$;
- the Racah–Wigner algebras $\mathcal{RW}^{(\lambda)}_\infty(\mathrm{sl}(2,\mathbb{C}))$ for $\mathrm{sl}(2,\mathbb{C})$, their reductions $\mathcal{RW}^{(\lambda)}_n(\mathrm{sl}(2,\mathbb{C}))$ of order n and central extensions $\widehat{\mathcal{RW}}^{(\lambda)}_n(\mathrm{sl}(2,\mathbb{C}))$ of $\mathcal{RW}^{(\lambda)}_n(\mathrm{sl}(2,\mathbb{C}))$.

The Poisson algebra $\mathcal{F}^{\mathbb{C}}(\mathbb{S}^1)$ may be decomposed with respect to the action of $\mathbb{C}\mathrm{Vect}(\mathbb{S}^1)$ into the sum of tensor modules of positive integer weights, the corresponding tensor operators $w^{(i)}_k$ $(i \geq 1)$ being equal to e^{i-1}_k so that the commutation relations

$$[w^{(i)}_m, w^{(j)}_n] = ((j-1)n - (i-1)m)\, w^{(i+j-2)}_{n+m}$$

hold. The algebra spanned by $w^{(i)}_n$, $n \in \mathbb{Z}$, $i \geq 2$ is closed and is called *w_∞-algebra* (or *Bakas algebra*) [23] as well as its central extension, which is obtained from $\widehat{\mathcal{F}}^{\mathbb{C}}(T^*\mathbb{S}^1)$.

Now we are interested in the deformations of the w_∞-algebra, when $\widehat{\mathcal{F}}^{\mathbb{C}}(T^*\mathbb{S}^1)$ deforms into $\widehat{\mathrm{DOP}}^{\mathbb{C}}_{[\cdot,\cdot]}(\mathbb{S}^1)$. Such deformations were found by C.N. Pope, X. Shen and L.J. Romans [24] (see also Ref. [25]), so the corresponding algebras are called *Pope–Shen–Romans algebras* (or *W_∞-algebras*). The explicit formulas for their generators were found by I. Bakas, B. Khesin and E. Kiritsis [8] so the embeddings of W_∞-algebras in $\widehat{\mathrm{DOP}}^{\mathbb{C}}_{[\cdot,\cdot]}(\mathbb{S}^1)$ are called *Bakas–Khesin–Kiritsis embeddings*.

First of all, let us construct, following Ref. [8], the so-called $W_{1+\infty}$-algebra, which is an algebra of $\mathbb{C}$vir-tensor operators in $\widehat{\mathrm{DOP}}^{\mathbb{C}}_{[\cdot,\cdot]}(\mathbb{S}^1)$. Its generators V^s_m are labelled by $m \in \mathbb{Z}$ and $s \geq 1$. Their explicit form is

$$V^s_m = -B(s) \sum_{k=1}^{s} \alpha^s_k C^{m+s-1}_{k-1} z^{m+s-k} D^{s-k}_z,$$

where

$$B(s) = \frac{2^{s-3}(s-1)!}{(2s-3)!!}, \quad \alpha^s_k = \frac{(2s-k-1)!}{[(s-k)!]^2}; \quad z = \exp(iq), \quad D_z = \frac{\partial}{\partial z}.$$

To describe commutation relations in $W_{1+\infty}$-algebra it is convenient to introduce some notations:

$$g^{ss'}_l(m,n;\mu) := \frac{1}{2(l+1)!} f^{ss'}_l(\mu) N^{ss'}_l(m,n),$$

$$f^{ss'}_l(\mu) := \sum_{k\geq 0} \frac{(-\frac{1}{2}-2\mu)_k(\frac{3}{2}+2\mu)_k(-\frac{1}{2}(l+1))_k(-\frac{1}{2}l)_k}{k!\,(-s+\frac{3}{2})_k(-s'+\frac{3}{2})_k(s+s'-l-\frac{3}{2})_k},$$

$$N^{ss'}_l(m,n) := \sum_{k=0}^{l+1} (-1)^k C^{l+1}_k (2s-l-2)_k [2s'-k-2]_{l+1-k} \times [s-1+m]_{l+1-k}[s'-1+n]_k,$$

where $(a)_k := a(a+1)\cdots(a+k-1)$, $[a]_k := a(a-1)\cdots(a-k+1)$. Then

$$[V^s_m, V^{s'}_n] = ((s'-1)m-(s-1)n)V^{s+s'-2}_{m+n} + c_s(m,\mu)\delta_{ss'}\delta(m+n) + \sum_{r\geq 0} g^{ss'}_{2r}(m,n;\mu)V^{s+s'-2r-2}_{m+n},$$

where $\mu = -\frac{1}{2}$,

$$c_s(m;-\tfrac{1}{2}) = c \cdot \frac{(m+s-1)!}{(m-s)!} \cdot \frac{2^{2(s-3)}[(s-1)!]^2}{(2s-1)!!\,(2s-3)!!}.$$

The $W_{1+\infty}$-algebra is not a deformation of Bakas algebra because it contains tensor operators of upper index 1. To construct the correct deformation (which is the "right" W_∞-algebra) one should transform the generators V^s_m in the following manner. Let us introduce new generators W^s_m ($m \in \mathbb{Z}$, $s \geq 2$) by the formulas

$$W^s_m = V^s_m + \frac{B(s)}{s-1} \sum_{k=1}^{s-1} (-1)^l \frac{(2s-2l-1)}{B(s-l)} \cdot \frac{(m+s-1)!}{(m+s-l-1)!} V^{s-1}_m$$

or, explicitly,

$$W^s_m = -\frac{B(s)}{s-1} \sum_{k=1}^{s-1} \beta^s_k C^{m+s-1}_{k-1} z^{m+s-k} D^{s-k}_z,$$

where $\beta_k^s = (2s-k-1)!/(s-k)!(s-k-1)!$, i.e. the W_∞-algebra is realized in $\mathrm{DOP}^{\mathbb{C}}_{[\cdot,\cdot]}(\mathbb{S}^1)_+$. The commutation relations in W_∞-algebra have the form

$$[W_m^s, W_n^{s'}] = ((s'-1)m-(s-1)n)W_{m+n}^{s+s'-2} + c_s(m;\mu)\delta_{ss'}\delta(m+n) + \sum_{r\geq 1} g_{2r}^{ss'}(m,n;\mu)W_{m+n}^{s+s'-2-2r},$$

where $\mu = 0$,

$$c_s(m,0) = \frac{c}{2}\cdot\frac{(m+s-1)!}{(m-s)!}\cdot\frac{2^{2(s-3)}s!(s-2)!}{(2s-1)!!(2s-3)!!}.$$

Let us consider following Refs. [6,7] the affine subspace $\mathcal{M}_n = \{A \in \mathrm{DOP}^{\mathbb{C}}(\mathbb{S}^1) : A = D^n + a_{n-1}(q)D^{n-1} + \cdots + a_1(q)D + a_0(q)\}$ in the algebra $\mathrm{DOP}^{\mathbb{C}}(\mathbb{S}^1)$. The tangent space $T_A(\mathcal{M}_n)$ of the manifold $\mathcal{M}_n$ at the point A may be identified with the space of all operators X from $\mathrm{DOP}^{\mathbb{C}}(\mathbb{S}^1)$ of the form $X = x_{n-1}(q)D^{n-1} + \cdots + x_0(q)$. The corresponding cotangent space $T_A^*(\mathcal{M}_n)$ may be identified with the space of all operators Y from $\mathrm{PDOP}^{\mathbb{C}}(\mathbb{S}^1)$ of the form $Y = y_{-1}(q)D^{-1} + \cdots + y_{1-n}(q)D^{1-n} + y_{-n}(q)D^{-n}$. The spaces $T_A^*(\mathcal{M}_n)$ and $T_A(\mathcal{M}_n)$ are paired by $\langle X, Y\rangle = \mathrm{Tr}(XY)$.

There is defined [6,7] a tensor operator field $V : T^*(\mathcal{M}_n) \mapsto T(\mathcal{M}_n)$, namely V_A, which maps $T_A^*(\mathcal{M}_n)$ to $T_A(\mathcal{M}_n)$, and is defined by the following formulas: $V_A(Y) = A(YA)_+ - (AY)_+A$, $Y \in T_A^*(\mathcal{M}_n)$, where $A \mapsto A_+$ is the natural projection of $\mathrm{PDOP}^{\mathbb{C}}(\mathbb{S}^1)$ to $\mathrm{DOP}^{\mathbb{C}}(\mathbb{S}^1)$.

The space $\mathcal{F}(\mathcal{M}_n)$ of functions on the manifold $\mathcal{M}_n$ possesses a structure of the Poisson algebra via $\{F, G\} = \langle V(dF), dG\rangle$ [6,7]. This Poisson algebra is called *Gelfand-Dickey algebra* and is denoted by $\mathrm{GD}_n^{\mathbb{C}}$. The center of the Gelfand-Dickey algebra $\mathrm{GD}_n^{\mathbb{C}}$ consists of constant functions, so the exact sequence

$$0 \longrightarrow \mathbb{C} \longrightarrow \mathrm{GD}_n^{\mathbb{C}} \longrightarrow \overline{\mathrm{GD}}_n^{\mathbb{C}} \longrightarrow 0$$

exists, where $\overline{\mathrm{GD}}_n^{\mathbb{C}}$ is the quotient $\mathrm{GD}_n^{\mathbb{C}}/\mathbb{C}$.

Let us now construct *the Radul mapping* from the Poisson algebra $\mathcal{F}(\mathrm{DOP}^{\mathbb{C}}_{[\cdot,\cdot]}(\mathbb{S}^1)^*)$ of the functions on the coadjoint module $\mathrm{DOP}^{\mathbb{C}}_{[\cdot,\cdot]}(\mathbb{S}^1)^*)$ of the Lie algebra $\mathrm{DOP}^{\mathbb{C}}_{[\cdot,\cdot]}(\mathbb{S}^1)$ onto the Poisson algebra $\mathrm{GD}_n^{\mathbb{C}}$. Namely, let us construct a mapping

$$H : \mathrm{DOP}^{\mathbb{C}}_{[\cdot,\cdot]}(\mathbb{S}^1) \mapsto \mathbb{C}\mathrm{Vect}(\mathcal{M}_n),$$

where the mapping $H_A : \mathrm{DOP}^{\mathbb{C}}_{[\cdot,\cdot]}(\mathbb{S}^1) \mapsto T_A(\mathcal{M}_n)$ is defined by the formulas [6,7]

$$H_A(B) = V_{(BA^{-1})_-}(A) = AB - (ABA^{-1})_+A;$$
$$B \in \mathrm{DOP}^{\mathbb{C}}_{[\cdot,\cdot]}(\mathbb{S}^1), \quad A_- = A - A_+.$$

It should be mentioned that

$$\mathrm{Ker}(H_A) = \{C \in \mathrm{DOP}^{\mathbb{C}}_{[\cdot,\cdot]}(\mathbb{S}^1) : C = BA, \quad A \in \mathrm{DOP}^{\mathbb{C}}(\mathbb{S}^1)\}.$$

We state that $\mathrm{Im}(H)$ is contained in $\mathrm{Im}(d)$, where $d : \mathrm{GD}_n^{\mathbb{C}} \mapsto \mathbb{C}\mathrm{Vect}(\mathcal{M}_n)$. Because $\mathrm{Ker}(d)$ consists of constant functions on $\mathcal{M}_n$ then H realizes a homomorphism of $\mathrm{DOP}^{\mathbb{C}}_{[\cdot,\cdot]}(\mathbb{S}^1)$ into the Lie algebra $\overline{\mathrm{GD}}^{\mathbb{C}}_n$. Such a homomorphism can be extended to a homomorphism of the Poisson algebra $\mathcal{F}(\mathrm{DOP}^{\mathbb{C}}_{[\cdot,\cdot]}(\mathbb{S}^1)^*)$ onto the Gelfand–Dickey algebra $\mathrm{GD}^{\mathbb{C}}_n$. Of course, one may consider a restriction of the Radul mapping $\mathcal{F}(\mathrm{DOP}^{\mathbb{C}}_{[\cdot,\cdot]}(\mathbb{S}^1)^*) \mapsto \mathrm{GD}^{\mathbb{C}}_n$ to the Poisson subalgebra $\mathcal{F}(\mathrm{DOP}^{\mathbb{C}}_{[\cdot,\cdot]}(\mathbb{S}^1)^*_+)$ of the Poisson algebra $\mathcal{F}(\mathrm{DOP}^{\mathbb{C}}(\mathbb{S}^1)^*)$, where $\mathrm{DOP}^{\mathbb{C}}(\mathbb{S}^1)_+$ is a subalgebra of $\mathrm{DOP}^{\mathbb{C}}(\mathbb{S}^1)$ of differential operators without free terms, $\mathrm{DOP}^{\mathbb{C}}_{[\cdot,\cdot]}(\mathbb{S}^1)_+$ is its commutator algebra, the Lie algebra of differential operators without free terms, and $\mathrm{DOP}^{\mathbb{C}}_{[\cdot,\cdot]}(\mathbb{S}^1)^*_+$ is the coadjoint module over this Lie algebra.

Now let us construct the so-called *Radul bundle* and the Lie algebra of its sections (*Radul algebra*). Let us consider a trivial bundle $\mathrm{Rad}^{\mathbb{C}}(\mathcal{M}_n)$ over $\mathcal{M}_n$ with fiber isomorphic to $\mathrm{DOP}^{\mathbb{C}}_{[\cdot,\cdot]}(\mathbb{S}^1)$. One may define the following commutator in the space of its sections: if $E, F \in \mathcal{O}(\mathrm{Rad}^{\mathbb{C}}(\mathcal{M}_n))$ then

$$[E,F]'(A) := [E(A),F(A)] + V_{(EA^{-1})_-}(F) - V_{(FA^{-1})_-}(E) \in \mathrm{Rad}^{\mathbb{C}}_A(\mathcal{M}_n),$$

$$V_X(E(A)) = (\partial/\partial t)E(A+tV_X(A))|_{t=0}, \quad A \in \mathcal{M}_n.$$

The algebra of sections of the bundle $\mathrm{Rad}^{\mathbb{C}}(\mathcal{M}_n)$ will be called the Radul algebra $\mathrm{Rad}_{\mathbb{C}}$. As it was stated in Ref. [7] the mappings H_A being glued together define a homomorphism of the Radul algebra $\mathrm{Rad}_{\mathbb{C}}$ onto the Lie algebra $\overline{\mathrm{GD}}^{\mathbb{C}}_n$.

The algebra $\mathrm{GD}^{\mathbb{C}}_n$ defined above is just the Gelfand–Dickey algebra $\mathrm{GD}(\mathrm{gl}(n,\mathbb{C}))$ for the Lie algebra $\mathrm{gl}(n,\mathbb{C})$. It is more convenient to consider the Gelfand–Dickey algebra $\mathrm{GD}(\mathrm{sl}(n,\mathbb{C}))$ for the Lie algebra $\mathrm{sl}(n,\mathbb{C})$. To do it one should consider the subspace $\mathcal{M}^{(0)}_n$ of $\mathcal{M}_n$ of operators A with $a_{n-1}=0$. Note that our representation may be considered for $\mathrm{GD}(\mathrm{sl}(n,\mathbb{C}))$ if one replaces $\mathrm{DOP}^{\mathbb{C}}_{[\cdot,\cdot]}(\mathbb{S}^1)$ by $\mathrm{DOP}^{\mathbb{C}}_{[\cdot,\cdot]}(\mathbb{S}^1)_+$.

As was remarked in Ref. [24], the generators W^s_m of the Pope–Shen–Romans W_∞-algebra with $1-s \le m \le s-1$ form a closed Lie algebra, which is called the *wedge subalgebra* of the W_∞-algebra. The generators W^2_{-1}, W^2_0, W^2_1 of the wedge subalgebra are just the generators of the Lie algebra $\mathrm{sl}(2,\mathbb{C})$. The space of the wedge subalgebra is identified as $\mathrm{sl}(2,\mathbb{C})$-module with the model $M_{\text{odd-dim}}(\mathrm{sl}(2,\mathbb{C}))$ of odd-dimensional representations of the Lie algebra $\mathrm{sl}(2,\mathbb{C})$ [26].

Let us analyse the algebraic structures related to the model $M_{\text{odd-dim}}(\mathrm{sl}(2,\mathbb{C}))$ more systematically. Let us consider the universal enveloping algebra $\mathcal{U}(\mathrm{sl}(2,\mathbb{C}))$ of the Lie algebra $\mathrm{sl}(2,\mathbb{C})$ and the ideal I_λ in it, generated by $K-\lambda$, where K is a Casimir operator for $\mathrm{sl}(2,\mathbb{C})$ (i.e. a quadratic element of the center $\mathcal{Z}(\mathcal{U}(\mathrm{sl}(2,\mathbb{C})))$ of the universal enveloping algebra $\mathcal{U}(\mathrm{sl}(2,\mathbb{C}))$. The quotient $\mathcal{U}(\mathrm{sl}(2,\mathbb{C}))/I_\lambda$ supplies the model $M_{\text{odd-dim}}(\mathrm{sl}(2,\mathbb{C}))$ by a structure of an associative algebra. Such an associative algebra will be called *the associative model algebra for the Lie algebra* $\mathrm{sl}(2,\mathbb{C})$ and will be denoted by $\mathrm{Md}^{(\lambda)}(\mathrm{sl}(2,\mathbb{C}))$.

The commutator algebra $\mathrm{Md}^{(\lambda)}_{[\cdot,\cdot]}(\mathrm{sl}(2,\mathbb{C}))$ of the associative model algebra $\mathrm{Md}^{(\lambda)}(\mathrm{sl}(2,\mathbb{C}))$ will be called *the model Lie algebra for the Lie algebra* $\mathrm{sl}(2,\mathbb{C})$ (or *Feigin algebra*, see Ref. [27]). The wedge subalgebra of the Pope–Shen–Romans W_∞-algebra is just the Feigin algebra $\mathrm{Md}^{(0)}_{[\cdot,\cdot]}(\mathrm{sl}(2,\mathbb{C}))$.

Let us now consider the Racah–Wigner algebras, their reductions of finite order and central extensions of such reductions for the Lie algebra $\mathrm{sl}(2,\mathbb{C})$. *The Racah–Wigner algebra* $\mathcal{RW}_{\infty}^{(\lambda)}(\mathrm{sl}(2,\mathbb{C}))$ *for the Lie algebra* $\mathrm{sl}(2,\mathbb{C})$ is the universal enveloping algebra $\mathcal{U}(\mathrm{Md}_{[\cdot,\cdot]}^{(\lambda)}(\mathrm{sl}(2,\mathbb{C})))$ of the model Lie algebra $\mathrm{Md}_{[\cdot,\cdot]}^{(\lambda)}(\mathrm{sl}(2,\mathbb{C}))$.

Definition 1. An associative algebra $\mathcal{RW}_{n}^{(\lambda)}(\mathrm{sl}(2,\mathbb{C}))$ is called *a reduced Racah–Wigner algebra of order n for the Lie algebra* $\mathrm{sl}(2,\mathbb{C})$ iff

(i) it admits a homomorphism onto $\mathrm{Md}^{(\lambda)}(\mathrm{sl}(2,\mathbb{C}))$;

(ii) $\mathcal{RW}_{n}^{(\lambda)}(\mathrm{sl}(2,\mathbb{C}))$ is generated by the direct sum $\pi_1\oplus\pi_2\oplus\cdots\oplus\pi_{n-1}\oplus\pi_n$ of the first n odd-dimensional representations of $\mathrm{sl}(2,\mathbb{C})$ (this direct sum is isomorphic to $\mathrm{sl}(n,\mathbb{C})$ as $\mathrm{sl}(2,\mathbb{C})$-module), the natural action of $\mathrm{sl}(2,\mathbb{C})$ being defined by $\mathrm{ad}(\pi_1)$;

(iii) $\mathcal{RW}_{n}^{(\lambda)}(\mathrm{sl}(2,\mathbb{C}))$ is isomorphic as $\mathrm{sl}(2,\mathbb{C})$-module to $S^{\cdot}(\mathrm{sl}(n,\mathbb{C}))$ ($S^{\cdot}(V)$ is the symmetric algebra over V).

Reduced Racah–Wigner algebras $\mathcal{RW}_{2}^{(\lambda)}(\mathrm{sl}(2,\mathbb{C}))$ of order 2 with non-homogeneous quadratic relations were considered in Ref. [28]. It was shown that $\mathcal{RW}_{2}^{(\lambda_1)}(\mathrm{sl}(2,\mathbb{C}))\simeq\mathcal{RW}_{2}^{(\lambda_2)}(\mathrm{sl}(2,\mathbb{C}))$ for arbitrary λ_1 and λ_2.

Proposition 1. *For any n, there exists a reduced Racah–Wigner algebra* $\mathcal{RW}_{n}^{(\lambda)}(\mathrm{sl}(2,\mathbb{C}))$ *of finite order n for the Lie algebra* $\mathrm{sl}(2,\mathbb{C})$.

Proof. Unfortunately, we do not know the direct algebraic proof of this proposition, which is, certainly, preferable. Let us formulate a quantum-field proof, which is based on the formalism of q_R-conformal field theory [28]. Namely, let us consider the set of q_R-affine currents, whose charges form the Lie algebra $\mathrm{sl}(n,\mathbb{C})$ (or, whose components form a q_R-affine Lie algebra) [28]. Let us consider the set of Casimir operators $K_1,\ldots,K_{n-1}$ and the higher spin fields $W^2(z),\ldots,W^n(z)$, which correspond to them in the operator algebra of q_R-affine currents. The algebra, generated by the components W^s_m ($1-s\le m\le s-1$; $W^s(z)=\sum_{m\in\mathbb{Z}}W^s_m z^{-s-m}$) is just the reduced Racah–Wigner algebra $\mathcal{RW}_{n}^{(\lambda)}(\mathrm{sl}(2,\mathbb{C}))$ of order n for the Lie algebra $\mathrm{sl}(2,\mathbb{C})$, where $\lambda=\frac{1}{4}(q_R^{-1}+3)(q_R^{-1}+1)$. □

The non-linear structure constants of the constructed reduced Racah–Wigner algebras $\mathcal{RW}_{n}^{(\lambda)}(\mathrm{sl}(2,\mathbb{C}))$ admit a deformation by the linear structure constants of the universal enveloping algebra $\mathcal{U}(\mathrm{sl}(n,\mathbb{C}))$ of the Lie algebra $\mathrm{sl}(n,\mathbb{C})$. One may introduce a new central element ρ and consider this deformation as a central extension of the reduced Racah–Wigner algebra. Such a central extension will be denoted by $\widehat{\mathcal{RW}}_{n}^{(\lambda)}(\mathrm{sl}(n,\mathbb{C}))$. If one considers the fixed value of central element ρ, i.e. the quotient of $\widehat{\mathcal{RW}}_{n}^{(\lambda)}(\mathrm{sl}(n,\mathbb{C}))$ with the ideal generated by $\rho-\rho_0$, then the limit $\rho_0\to\infty$ inverts the obtained quotient into the universal enveloping algebra $\mathcal{U}(\mathrm{sl}(n,\mathbb{C}))$. So one may consider the algebras $\widehat{\mathcal{RW}}_{n}^{(\lambda)}(\mathrm{sl}(2,\mathbb{C}))$ to be deformations of $\mathcal{U}(\mathrm{sl}(n,\mathbb{C}))$.

2. Gervais–Matsuo differential W-geometry and W-symmetries of a second quantized free string

In this paragraph we shall work with the following objects [1] (see also Refs. [13,29–31]):

(i) Q (or the dual Q^*)—the space of external degrees of freedom of a string. The coordinates x_n^μ on Q are the Taylor coefficients of functions $x^\mu(z)$, which determine the world-sheet of a string in a complexified target space.

(ii) $M(\mathrm{Vir})$—the space of internal degrees of a freedom of a string; *the flag manifold of the Virasoro–Bott group* Vir; the homogeneous space $\mathrm{Diff}_+(\mathbb{S}^1)/\mathbb{S}^1$ ($\mathrm{Diff}_+(\mathbb{S}^1)$ is the group of diffeomorphisms of a circle $\mathbb{S}^1$ preserving orientation); this space is identified via the Kirillov construction [13, Kirillov] with

(iii) S—*the class of univalent functions* $f(z)$ in the unit complex disc D_+ ($D_+ = \{z \in \mathbb{C}\colon |z| \leq 1\}$) such that $f(0) = 0$, $f'(0) = 1$, $f'(e^{it}) \neq 0$; the natural coordinates on S are coefficients c_k of the Taylor expansion of a univalent function $f(z)$: $f(z) = z + c_1 z^2 + c_2 z^3 + \cdots + c_{n-1} z^n + c_n z^{n+1} + \cdots$.

(iv) C—*the universal deformation of a complex disc* with $M(\mathrm{Vir})$ as a base and with fibers isomorphic to D_+. The coordinates on C are $z, c_1, c_2, \ldots, c_n, \ldots$, where c_k are coordinates on the base and z is a coordinate in the fibers.

(v) $M(\mathrm{Vir}) \cdot Q^*$—the space of both external and internal degrees of freedom of a string, the same as the bundle over $M(\mathrm{Vir})$ associated with $p : C \mapsto M(Vir)$, whose fibers are $\mathrm{Map}(C/M(\mathrm{Vir}); \mathbb{C}^n)^*$—linear spaces dual to spaces of mappings of fibers of $p : C \mapsto M(\mathrm{Vir})$ into $\mathbb{C}^n$.

2.1. Elements of Gervais–Matsuo W-geometry

In this subsection we discuss

- Elements of the Gervais–Matsuo differential W-geometry: classical Toda fields in the complex analogue of Frenet theory;
- the mapping $\mathrm{DOP}^{\mathbb{C}}_{[\cdot,\cdot]}(\mathbb{S}^1)^{\mathrm{reg}}_+ \mapsto \mathbb{C}\mathrm{Vect}(M(\mathrm{Vir}) \cdot Q^*)$;
- the Gervais–Matsuo Lie quasi(pseudo)algebra $\mathcal{GM}_n^{\mathbb{C}}$ of classical (infinitesimal) W-transformations;
- the Gervais–Matsuo Poisson algebra $\mathrm{GM}_n^{\mathbb{C}}$ and the monomorphism $\mathrm{GM}_n^{\mathbb{C}} \mapsto \mathrm{GD}(\mathrm{sl}(n, \mathbb{C}))$;
- the infinite dimensional geometry of the flag manifold $M(\widehat{\mathrm{DOP}}_{[\cdot,\cdot]}(\mathbb{S}^1)_+) \simeq M(W^r_\infty)$ ($W_\infty = (W^r_\infty)^{\mathbb{C}}$) for the Lie algebra $\widehat{\mathrm{DOP}}_{[\cdot,\cdot]}(\mathbb{S}^1)_+$ or for the real form W^r_∞ of the Pope–Shen–Romans W_∞-algebra.

Let us consider the world-sheet of a string $\boldsymbol{x} : D_+ \mapsto \mathbb{C}^n$ ($\boldsymbol{x} = \{x^\mu = x^\mu(z)\}$). By using the complex version of Frenet theory [12] one may introduce the associated mappings $\mathrm{Gr}_k\, \boldsymbol{x} : D_+ \mapsto \mathrm{Gr}_k(\mathbb{C}^n)$, where $\mathrm{Gr}_k(\mathbb{C}^n)$ is the Grassmannian of k-dimensional planes in $\mathbb{C}^n$, or the associated mapping $\mathrm{Fl}\, \boldsymbol{x} : D_+ \mapsto \mathrm{Fl}(\mathbb{C}^n)$, where $\mathrm{Fl}(\mathbb{C}^n)$ is the space of complete flags in $\mathbb{C}^n$. If we consider the homogeneous coordinates (equivalently, coordinates in the projective space $\mathbb{CP}^n$, then *the classical Toda fields* on D_+ will be identified with Kähler potentials of the images of $\mathrm{Gr}_{k+1}\,\boldsymbol{x}(z)$ in $\mathrm{Gr}_{k+1}(\mathbb{C}^{n+1})$.

It is convenient to introduce the so-called *homogeneous KP-coordinates* for the mapping $\boldsymbol{x} : D_+ \mapsto \mathbb{CP}^n$. Indeed, let us consider the functions $x^\mu([z])$, $[z] = [z^{(0)}, z^{(1)} = z, z^{(2)}, \dots z^{(n)}, \dots]$ such that $D_z^l \boldsymbol{x}([z]) = \partial \boldsymbol{x}([z])/\partial z^{(l)}$ and $\boldsymbol{x}([z]) = \boldsymbol{x}(z)$ for $z^{(2)}, z^{(3)}, \dots z^{(n)}, \dots = 0$, $z = z^{(1)}/z^{(0)}$. So the homogeneous KP-coordinates $[z^{(0)}, z^{(1)}, \dots, z^{(n)}]$ may be regarded as coordinates in the complex projective space $\mathbb{CP}^n$ defined in the neighborhood of the world-sheet of a string.

Let now $\mathrm{DOP}^{\mathbb{C}}(\mathbb{S}^1)^{\mathrm{reg}}$ be a subalgebra of $\mathrm{DOP}^{\mathbb{C}}(\mathbb{S}^1)$ that consists of operators possessing a regular continuation to the unit complex disc D_+; $\mathrm{DOP}^{\mathbb{C}}(\mathbb{S}^1)^{\mathrm{reg}}_+$ the intersection of $\mathrm{DOP}^{\mathbb{C}}(\mathbb{S}^1)^{\mathrm{reg}}$ with $\mathrm{DOP}^{\mathbb{C}}(\mathbb{S}^1)_+$; $\mathrm{DOP}^{\mathbb{C}}_{[\cdot,\cdot]}(\mathbb{S}^1)^{\mathrm{reg}}$ and $\mathrm{DOP}^{\mathbb{C}}_{[\cdot,\cdot]}(\mathbb{S}^1)^{\mathrm{reg}}_+$ the corresponding commutator Lie algebras.

It should be mentioned that $\mathrm{DOP}^{\mathbb{C}}_{[\cdot,\cdot]}(\mathbb{S}^1)^{\mathrm{reg}}_+$ acts naturally by differential operators on the space Q^*, considered as the space of sections of a trivialized bundle over D_+ with fibers isomorphic to $\mathbb{C}^n$). One of the results of J.-L. Gervais and Y. Matsuo was that this action is linearized (i.e. becomes an action by vector fields) in KP-coordinates.

Proposition 2. *The action of the Lie algebra* $\mathrm{DOP}^{\mathbb{C}}_{[\cdot,\cdot]}(\mathbb{S}^1)^{\mathrm{reg}}_+$ *by differential operators on* Q^* *(as the space of sections of a trivialized bundle over* D_+ *with fibers isomorphic to* $\mathbb{C}^n$*) may be extended to the action on* $M(\mathrm{Vir}) \cdot Q^*$ *(as the space of sections of a trivialized bundle over* C *with fibers isomorphic to* $\mathbb{C}^n$*) by vector fields.*

Proof. One only needs to mention that the KP-coordinate system is a homogeneous version of the coordinates on the universal deformation C of a complex disc D_+, which are easily expressed via the standard coordinates $z, c_1, c_2, \dots c_n, \dots$. □

Let us now consider more systematically the structure of the action of the Lie algebra $\mathrm{DOP}^{\mathbb{C}}_{[\cdot,\cdot]}(\mathbb{S}^1)^{\mathrm{reg}}_+$ on Q^* by differential operators.

It is rather reasonable, following J.-L. Gervais and Y. Matsuo, to restrict ourselves to consideration of differential operators of order less than or equal to n. It means that we shall deal with the quotient $\mathrm{DOP}^{\mathbb{C}}_{[\cdot,\cdot]}(\mathbb{S}^1)^{\mathrm{reg}}_+/\mathrm{DOP}^{\mathbb{C}}_{[\cdot,\cdot]}(\mathbb{S}^1)^{\mathrm{reg}}_{\geq n+1}$ of the Lie algebra $\mathrm{DOP}^{\mathbb{C}}_{[\cdot,\cdot]}(\mathbb{S}^1)^{\mathrm{reg}}_+$ of regular differential operators without free terms by its subalgebra $\mathrm{DOP}^{\mathbb{C}}_{[\cdot,\cdot]}(\mathbb{S}^1)^{\mathrm{reg}}_{\geq n+1}$ of regular differential operators without free terms, which do not contain terms with D^k $(1 \leq k \leq n)$, the commutator algebra of the associative algebra $\mathrm{DOP}^{\mathbb{C}}(\mathbb{S}^1)^{\mathrm{reg}}_{\geq n+1}$ of such operators. Of course, such a quotient is not a Lie algebra, because the subalgebra $\mathrm{DOP}^{\mathbb{C}}_{[\cdot,\cdot]}(\mathbb{S}^1)^{\mathrm{reg}}_{\geq n+1}$ is not an ideal in $\mathrm{DOP}^{\mathbb{C}}_{[\cdot,\cdot]}(\mathbb{S}^1)^{\mathrm{reg}}_+$. Nevertheless, the exact sequence

$$\begin{aligned} 0 &\longrightarrow \mathrm{DOP}^{\mathbb{C}}_{[\cdot,\cdot]}(\mathbb{S}^1)^{\mathrm{reg}}_{\geq n+1} \longrightarrow \mathrm{DOP}^{\mathbb{C}}_{[\cdot,\cdot]}(\mathbb{S}^1)^{\mathrm{reg}}_+ \\ &\longrightarrow \mathrm{DOP}^{\mathbb{C}}_{[\cdot,\cdot]}(\mathbb{S}^1)^{\mathrm{reg}}_+/\mathrm{DOP}^{\mathbb{C}}_{[\cdot,\cdot]}(\mathbb{S}^1)^{\mathrm{reg}}_{\geq n+1} \longrightarrow 0 \end{aligned}$$

may be split. The splitting identifies the quotient$\mathrm{DOP}^{\mathbb{C}}_{[\cdot,\cdot]}(\mathbb{S}^1)^{\mathrm{reg}}_+/\mathrm{DOP}^{\mathbb{C}}_{[\cdot,\cdot]}(\mathbb{S}^1)^{\mathrm{reg}}_{\geq n+1}$ with the subspace $\mathrm{DOP}^{\mathbb{C}}(\mathbb{S}^1)^{\mathrm{reg}}_{+;\leq n}$ of regular differential operators of order less than or equal to n without free terms in the Lie algebra $\mathrm{DOP}^{\mathbb{C}}_{[\cdot,\cdot]}(\mathbb{S}^1)^{\mathrm{reg}}_+$.

The object that elements of $\mathrm{DOP}^{\mathbb{C}}(\mathbb{S}^1)^{\mathrm{reg}}_{+;\le n}$ form is in fact a Lie quasialgebra (in the terminology of I. Batalin [32]) or a Lie pseudoalgebra (in the terminology of M.V. Karasev and V.P. Maslov [33]). Namely, the commutator of two elements of $\mathrm{DOP}^{\mathbb{C}}(\mathbb{S}^1)^{\mathrm{reg}}_{+;\le n}$ acting on Q^* (*classical (infinitesimal) W-transformations* [11]) may be expressed via other elements with coefficients, which are extrinsic invariants of the world-sheet of a string (curvature, torsions and their derivatives). So the structure functions of classical (infinitesimal) W-transformations are functions on the space Q^*, the object, on which classical (infinitesimal) W-transformations act. Therefore, classical (infinitesimal) W-transformations form a Lie quasi(pseudo)algebra, which will be called *Gervais–Matsuo quasi(pseudo)algebra* and will be denoted by $\mathcal{GM}^{\mathbb{C}}_n$.

It should be noted (concerning the possible construction of classical finite W-transformations) that Lie quasi(pseudo)algebras are infinitesimal objects for Lie quasi(pseudo)-groups of transformations in the finite dimensional case [32,33]. Fixing a point on a manifold, on which the Lie quasi(pseudo)group of transformations acts, one can introduce a structure of a loop on the underlying space of the Lie quasi(pseudo)group. In our case this loop will be just the loop of the homogeneous space

$$\mathrm{EXP}(\mathrm{DOP}^{\mathbb{C}}_{[\cdot,\cdot]}(\mathbb{S}^1)^{\mathrm{reg}}_{+})/\mathrm{EXP}(\mathrm{DOP}^{\mathbb{C}}_{[\cdot,\cdot]}(\mathbb{S}^1)^{\mathrm{reg}}_{\ge n+1})$$

defined by $\mathrm{EXP}(\mathrm{DOP}^{\mathbb{C}}(\mathbb{S}^1)^{\mathrm{reg}}_{+;\le n})$ to be a splitting of the exact sequence

$$\begin{aligned}0 &\longrightarrow \mathrm{EXP}(\mathrm{DOP}^{\mathbb{C}}_{[\cdot,\cdot]}(\mathbb{S}^1)^{\mathrm{reg}}_{\ge n+1}) \longrightarrow \mathrm{EXP}(\mathrm{DOP}^{\mathbb{C}}_{[\cdot,\cdot]}(\mathbb{S}^1)^{\mathrm{reg}}_{+})\\ &\longrightarrow \mathrm{EXP}(\mathrm{DOP}^{\mathbb{C}}_{[\cdot,\cdot]}(\mathbb{S}^1)^{\mathrm{reg}}_{+})/\mathrm{EXP}(\mathrm{DOP}^{\mathbb{C}}_{[\cdot,\cdot]}(\mathbb{S}^1)^{\mathrm{reg}}_{\ge n+1}) \longrightarrow 0\end{aligned}$$

via the *Sabinin construction* [34]. Nevertheless, the infinite dimensional groups

$$\mathrm{EXP}(\mathrm{DOP}^{\mathbb{C}}_{[\cdot,\cdot]}(\mathbb{S}^1)^{\mathrm{reg}}_{\ge n+1}) \quad \text{and} \quad \mathrm{EXP}(\mathrm{DOP}^{\mathbb{C}}_{[\cdot,\cdot]}(\mathbb{S}^1)^{\mathrm{reg}}_{+})$$

(as we know) are not constructed and, therefore, the classical finite W-transformations are not introduced; so the Sabinin construction may be considered only as an explanation of the appearance of the Gervais–Matsuo quasi(pseudo)algebra. However, even though the Lie quasi(pseudo)group of classical finite W-transformations is not defined (though the author thinks that it is possible) and, hence, the corresponding system of isotopic loops on it cannot be considered, the infinitesimal objects of such loops, the so called *Mikheev–Sabinin multialgebras* [35] may be derived easily from the Gervais–Matsuo quasi(pseudo)algebra $\mathcal{GM}^{\mathbb{C}}_n$.

It should be mentioned that Q^* is a symplectic manifold and, therefore, one may assign to the generators of the Gervais–Matsuo quasi(pseudo)algebra $\mathcal{GM}^{\mathbb{C}}_n$ their Hamiltonians, which are just functions of classical Toda fields. The structure of the Lie quasi(pseudo)algebra induces the structure of a Poisson algebra on Hamiltonians. This algebra will be called the *Gervais–Matsuo Poisson algebra* and will be denoted by $\mathrm{GM}^{\mathbb{C}}_n$. There exists a very simple but remarkable fact, which we prefer to formulate as a proposition.

Proposition 3. *There exists a monomorphism of Poisson algebras*

$$\mathrm{GM}_n^{\mathbb{C}} \mapsto \mathrm{GD}(\mathrm{sl}(n,\mathbb{C}))\,.$$

Therefore, the Gervais–Matsuo algebra $\mathrm{GM}_n^{\mathbb{C}}$ is just a "regular" part of the Gelfand–Dickey algebra $\mathrm{GD}(\mathrm{sl}(n,\mathbb{C}))$.

The significance of this monomorphism is explained by the fact that the *Gervais–Matsuo differential W-geometry may be (at least, partially) generalized, with an arbitrary Kähler manifold as a complexified target space*, because basically it depends only on complex Frenet theory (see [11,36]), *so one can construct analogues of the Gervais–Matsuo quasi(pseudo)algebra $\mathcal{GM}_n^{\mathbb{C}}$ of classical (infinitesimal) W-transformations as well as of the Gervais–Matsuo Poisson algebra $\mathrm{GM}_n^{\mathbb{C}}$ for an arbitrary Kähler manifold.*

It is not less remarkable fact that the *Gervais–Matsuo differential W-geometry turned out to be deeply connected with Sabinin's program of "nonlinear geometric algebra" [37] and the Weinstein–Karasev–Maslov approach to nonlinear Poisson brackets [33]*; so from the point of view of algebraic geometry, the Gervais–Matsuo W-geometry may be regarded as *a penetration of nonassociative algebra into the theory of embeddings of algebraic curves into Kähler varieties* (this is somewhat reminiscent of Yu.I. Manin's book [38]).

Let us now return to our main subject. As we saw above, the subalgebra W_∞^{reg} of the W_∞-algebra, generated by V_m^s, $m \geq 0$ acts by vector fields on the space $M(\mathrm{Vir})\cdot Q^*$. Nevertheless, we want to obtain the action of the whole W_∞-algebra (or $\widehat{\mathrm{DOP}}^{\mathbb{C}}_{[\cdot,\cdot]}(\mathbb{S}^1)_-$) instead of W_∞^{reg} (or $\mathrm{DOP}^{\mathbb{C}}_{[\cdot,\cdot]}(\mathbb{S}^1)_+^{\mathrm{reg}}$). So we should enlarge the space of the internal symmetries of a string by new degrees of freedom. What we are going to do is to consider the flag manifold $M(\widehat{\mathrm{DOP}}_{[\cdot,\cdot]}(\mathbb{S}^1)_+)$ for the Lie algebra $\widehat{\mathrm{DOP}}_{[\cdot,\cdot]}(\mathbb{S}^1)_+$ (or, what is just the same, the flag manifold $M(W_\infty^r)$ of the real form W_∞^r of the Pope–Shen–Romans W_∞-algebra), to consider the squashed product $M(W_\infty^r)\cdot Q^*$ and to justify that W_∞-algebra acts on $M(W_\infty^r)\cdot Q^*$ by vector fields. To perform this program we need a detailed description of the flag manifold $M(W_\infty^r)$.

The flag manifold $M(W_\infty^r)$ may be defined as a symplectic leaf of the Poisson manifold $(W_\infty^r)^*$, the coadjoint module for the Lie algebra W_∞^r (cf. [4]). The tangent space of $M(W_\infty^r)$ at the initial point may be identified with the quotient $W_\infty/W_\infty^{\mathrm{reg}}$, so the flag manifold $M(W_\infty^r)$ is an almost complex manifold.

Proposition 4. *The almost complex structure on the flag manifold $M(W_\infty^r)$ is integrable.*

Proof. First of all, the almost complex structure on $M(W_\infty^r)$ is formally integrable. To prove that it is really integrable one needs to construct an almost complex embedding of $M(W_\infty^r)$ into some infinite dimensional complex manifold. The standard manifold for such purposes is an infinite dimensional Grassmannian, e.g. one of the subspaces in W_∞ (one of which is W_∞^{reg}). □

Being the symplectic leaf of the Poisson manifold $(W_\infty^r)^*$, the flag space $M(W_\infty^r)$

possesses an infinite dimensional family of symplectic structures $\omega_{\boldsymbol{h},c}$, where c corresponds to the central charge of W^r_∞ and $\boldsymbol{h} = (h_2, h_3, \dots, h_n, \dots)$ corresponds to a character of the subalgebra W^{reg}_∞. Coupling with the complex structure 2-forms $\omega_{\boldsymbol{h},c}$ defines an infinite dimensional family of (pseudo)Kähler metrics on the flag manifold $M(W^r_\infty)$.

Each (pseudo)Kähler metric defines a prequantization bundle $E_{\boldsymbol{h},c}(M(W^r_\infty))$ over $M(W^r_\infty)$, which is a Hermitean line bundle with the action of the Lie algebra W_∞ by covariant derivatives with curvature form $2\pi\omega_{\boldsymbol{h},c}$.

The Verma modules over the Pope–Shen–Romans W_∞ algebra are realized in the Fock spaces $F(M(W^r_\infty), E_{\boldsymbol{h},c})$. Recall that the Fock space of a pair (M, E) is the space dual to the space of sections of the bundle E^* over M [13, Juriev]. It should be mentioned that Verma modules over $\mathrm{DOP}^{\mathbb{C}}_{[\cdot,\cdot]}(\mathbb{S}^1)$ were investigated recently by V. Kac and A. Radul [9].

2.2. *W-symmetries and string field theory*

In this subsection we will discuss

- W-symmetries of a second quantized free string (flat background): W-ghosts, W-differential Banks–Peskin forms, Siegel W-string fields and W-BRST-operator in them;
- W-string Gauss–Manin connection and W-string Kostant–Blattner–Sternberg pairings;
- geometrical non-cancellation of Bowick–Rajeev W-anomaly—absence of Bowick–Rajeev W-vacua and gauge-invariant Siegel W-string fields;
- operator cancellation of Bowick–Rajeev W-anomaly—W_N-algebras.

The flag manifold $M(W^r_\infty)$ for the W-algebra W^r_∞ admits an embedding into the infinite dimensional analogue of the classical symmetric domain of type I [39]. Such an embedding is defined by the mapping $W^r_\infty \mapsto \mathrm{gl}(\infty)$, where $\mathrm{gl}(\infty)$ is the Lie algebra of linear operators in the space $\mathcal{F}(\mathbb{S}^1)$ of functions on the circle $\mathbb{S}^1$. Since the infinite dimensional classical symmetric domain of type I admits a representation as a space of complex structures on $\mathcal{F}(\mathbb{S}^1) = \mathrm{Map}(\mathbb{S}^1, \mathbb{R})$ as well as on $\mathrm{Map}(\mathbb{S}^1, \mathbb{R}^n)$, we obtain such a representation for the flag manifold $M(W^r_\infty)$ too (this representation is analogous to one of M. Bowick and S. Rajeev for the flag manifold $M(\mathrm{Vir})$ for the Virasoro–Bott group [40]). Thus we have constructed the squash product $M(W^r_\infty) \cdot Q^*$.

Proposition 5. *The mapping* $W^{\mathrm{reg}}_\infty \mapsto \mathbb{C}\mathrm{Vect}(M(\mathrm{Vir}) \cdot Q^*)$ *is extended to the mapping* $W_\infty \mapsto \mathbb{C}\mathrm{Vect}(M(W^r_\infty) \cdot Q^*)$.

This proposition is a specialization of a standard fact of the theory of inductions to our infinite dimensional case.

Now we should mention that the Nambu-Goto action (the Kähler potential on Q^*) defines the bundle $E_{\mathrm{NG}}(Q^*)$, which may be lifted to the bundle $E_{\mathrm{NG}}(M(W^r_\infty) \cdot Q^*)$; this procedure defines the first cohomology class $H^1(W_\infty; \mathcal{O}(M(W^r_\infty) \cdot Q^*))$ of the Lie algebra W_∞ with coefficients in *"classical string fields"* $\mathcal{O}(M(W^r_\infty) \cdot Q^*)$ (cf. [1], see also [30]).

The Hermitean line bundle $E_{\boldsymbol{h},c}(M(W^r_\infty))$ over the flag manifold $M(W^r_\infty)$ may be lifted to the bundle $E_{\boldsymbol{h},c}(M(W^r_\infty)\cdot Q^*)$ over the space $M(W^r_\infty)\cdot Q^*$, in which the action of the Pope–Shen–Romans algebra W_∞ with non-trivial central charge is defined. One may also involve the first cohomology class of W_∞ in such an action. That means that one should consider the bundle $\tilde{E}_{\boldsymbol{h},c}(M(W^r_\infty)\cdot Q^*)$, the tensor product of $E_{\boldsymbol{h},c}(M(W^r_\infty)\cdot Q^*)$ and $E_{\mathrm{NG}}(M(W^r_\infty)\cdot Q^*)$, instead of $E_{\boldsymbol{h},c}(M(W^r_\infty)\cdot Q^*)$.

The Fock space $F(M(W^r_\infty)\cdot Q^*), \tilde{E}_{\boldsymbol{h},c}(M(W^r_\infty)\cdot Q^*))$ over the pair $(M(W^r_\infty)\cdot Q^*), \tilde{E}_{\boldsymbol{h},c}(M(W^r_\infty)\cdot Q^*))$ is just *"the configuration space for a second quantized free string without ghosts after accounting for W-symmetries"* (cf. [1], see also [29]).

Our next purpose is to introduce W-ghosts in order to consider W-differential Banks–Peskin forms, Siegel W-string fields, the W-BRST-operator in them and, thus, to construct *"the configuration space for a second quantized free string with ghosts after accounting for W-symmetries"* in a way analogous to that of Ref. [1] (see also [29]).

Unfortunately, even the first step of this program cannot be performed. The problem is that one cannot construct correctly the semi-infinite forms for the Pope–Shen–Romans W_∞-algebra (or for $\widehat{\mathrm{DOP}}^{\mathbb{C}}_{[\cdot,\cdot]}(\mathbb{S}^1)_+$). So we have to restrict our geometric picture to a finite order n. Namely, in view of the (real) Radul mapping $\mathcal{F}((W^r_\infty)^*)\mapsto \mathrm{GD}(\mathrm{sl}(n,\mathbb{R}))$ one may consider the flag manifold $M(\mathrm{GD}(\mathrm{sl}(n,\mathbb{R})))$ for the Gelfand–Dickey algebra instead of the flag manifold $M(W_\infty)$ for the W_∞-algebra. The flag manifold $M(\mathrm{GD}(\mathrm{sl}(n,\mathbb{R})))$ may be defined as a symplectic leaf of the Gelfand–Dickey algebra $\mathrm{GD}(\mathrm{sl}(n,\mathbb{R}))$. The flag manifold $M(\mathrm{GD}(\mathrm{sl}(n,\mathbb{R}))$ is a complex manifold as well as $M(W^r_\infty)$ and possesses (at least, one [4]) n-parametric family of symplectic structures $\omega_{\boldsymbol{h},c}$, where $\boldsymbol{h}=(h_2,\dots h_n)$. Coupled with the complex structure these 2-forms define an n-parametric family of (pseudo)Kähler metrics on the flag manifold $M(\mathrm{GD}(\mathrm{sl}(n,\mathbb{R})))$; each (pseudo)Kähler metric defines a prequantization bundle $E_{\boldsymbol{h},c}(M(\mathrm{GD}(\mathrm{sl}(n,\mathbb{R}))))$ over $M(\mathrm{GD}(\mathrm{sl}(n,\mathbb{R})))$.

Proposition 6. *The mapping $W_\infty \mapsto \mathbb{C}\mathrm{Vect}(M(W^r_\infty)\cdot Q^*)$ may be reduced to the mapping $W_\infty\mapsto \mathbb{C}\mathrm{Vect}(M(\mathrm{GD}(\mathrm{sl}(n,\mathbb{R})))\cdot Q^*)$.*

Proof. It is an easy consequence of the existence of the Radul mapping. □

One may perform for $M(\mathrm{GD}(\mathrm{sl}(n,\mathbb{R})))\cdot Q^*$ all geometric procedures described above for $M(W^r_\infty)\cdot Q^*$. Namely, the bundle $E_{\mathrm{NG}}(Q^*)$ may be lifted to the bundle $E_{\mathrm{NG}}(M(\mathrm{GD}(\mathrm{sl}(n,\mathbb{R})))\cdot Q^*)$; the Hermitean line bundle $E_{\boldsymbol{h},c}(M(\mathrm{GD}(\mathrm{sl}(n,\mathbb{R}))))$ may be lifted to the bundle $E_{\boldsymbol{h},c}(M(\mathrm{GD}(\mathrm{sl}(n,\mathbb{R})))\cdot Q^*)$; finally, one may consider the bundle $\tilde{E}_{\boldsymbol{h},c}(M(\mathrm{GD}(\mathrm{sl}(n,\mathbb{R})))\cdot Q^*)$ as the tensor product of two previous ones. It is natural to consider the Fock space $F(M(\mathrm{GD}(\mathrm{sl}(n,\mathbb{R})))\cdot Q^*, \tilde{E}_{\boldsymbol{h},c}(M(\mathrm{GD}(\mathrm{sl}(n,\mathbb{R})))\cdot Q^*))$ as *"the reduced configuration space for a second quantized free string without ghosts after accounting for W-symmetries"*.

But before we go on to consider the corresponding ghosts, it is reasonable to consider the structure of the action of the Lie algebra $W^{\mathbb{C}}_\infty$ on $M(\mathrm{GD}(\mathrm{sl}(n,\mathbb{R})))\cdot Q^*$ more systematically.

It seems that it is very convenient to restrict ourselves (as above) to consideration of differential operators of order less than or equal to n. It means that we shall deal with the quotient $\widehat{\mathrm{DOP}}^{\mathbb{C}}_{[\cdot,\cdot]}(\mathbb{S}^1)_+/\widehat{\mathrm{DOP}}^{\mathbb{C}}_{[\cdot,\cdot]}(\mathbb{S}^1)_{\geq n+1}$ of the central extension of the Lie algebra $\mathrm{DOP}^{\mathbb{C}}_{[\cdot,\cdot]}(\mathbb{S}^1)_+$ of differential operators without free terms by the central extension of its subalgebra $\mathrm{DOP}^{\mathbb{C}}_{[\cdot,\cdot]}(\mathbb{S}^1)_{\geq n+1}$ of differential operators without free terms, which do not contain terms with D^k $(1 \leq k \leq n)$. The corresponding exact sequence

$$\begin{aligned} 0 &\longrightarrow \widehat{\mathrm{DOP}}^{\mathbb{C}}_{[\cdot,\cdot]}(\mathbb{S}^1)_{\geq n+1} \longrightarrow \widehat{\mathrm{DOP}}^{\mathbb{C}}_{[\cdot,\cdot]}(\mathbb{S}^1)_+ \\ &\longrightarrow \widehat{\mathrm{DOP}}^{\mathbb{C}}_{[\cdot,\cdot]}(\mathbb{S}^1)_+/\widehat{\mathrm{DOP}}^{\mathbb{C}}_{[\cdot,\cdot]}(\mathbb{S}^1)_{\geq n+1} \longrightarrow 0 \end{aligned}$$

may be split. The splitting map identifies the quotient with the subspace $\mathrm{DOP}^{\mathbb{C}}(\mathbb{S}^1)_{+;\leq n}$ of differential operators of order less than or equal to n without free terms in the Lie algebra $\mathrm{DOP}^{\mathbb{C}}_{[\cdot,\cdot]}(\mathbb{S}^1)_+$. The object that is formed by elements of $\mathrm{DOP}^{\mathbb{C}}(\mathbb{S}^1)_{+;\leq n}$ is a Lie quasi(pseudo)algebra, which will be called *enlarged Gervais–Matsuo quasi(pseudo)-algebra* and will be denoted $\mathcal{GM}^{\mathbb{C}}_{n;(\mathrm{enl})}$. the Gervais–Matsuo quasi(pseudo)algebra may be comprehended as "slightly more than one-half of" the enlarged Gervais–Matsuo quasi(pseudo)algebra. It should be mentioned that the enlarged Gervais–Matsuo quasi-(pseudo)algebra is embedded in $\widehat{\mathrm{DOP}}^{\mathbb{C}}_{[\cdot,\cdot]}(\mathbb{S}^1)_+$ rather than in $\mathrm{DOP}^{\mathbb{C}}_{[\cdot,\cdot]}(\mathbb{S}^1)_+$ so we have obtained a central extension of the enlarged Gervais–Matsuo quasi(pseudo)algebra, which will be denoted by $\widehat{\mathcal{GM}}^{\mathbb{C}}_{n;(\mathrm{enl})}$. If the enlarged Gervais–Matsuo quasi(pseudo)-algebra $\mathcal{GM}^{\mathbb{C}}_{n;(\mathrm{enl})}$ acts on $M(\mathrm{GD}(\mathrm{sl}(n,\mathbb{R}))) \cdot Q^*$, then its central extension $\widehat{\mathcal{GM}}^{\mathbb{C}}_{n;(\mathrm{enl})}$ acts on the line bundles over it.

Now we are able to introduce W-ghosts (related to the central extended enlarged Gervais–Matsuo quasi(pseudo)algebra), to consider the corresponding W-differential Banks–Peskin forms and Siegel W-fields, the W-BRST-operator in them and, thus, to construct *"the reduced configuration space for a second quantized free string with ghosts after accounting for W-symmetries"*. Namely, *W-ghosts* may be identified with elements of the Lie quasi(pseudo)algebra $\mathcal{GM}^{\mathbb{C}}_{n;(\mathrm{enl})}$ acting on the manifold $M(\mathrm{GD}(\mathrm{sl}(n,\mathbb{R}))) \cdot Q^*$ by vector fields, W-antighosts are dual 1-forms on $M(\mathrm{GD}(\mathrm{sl}(n,\mathbb{R}))) \cdot Q^*$. *$W$-differential Banks–Peskin forms* are just differential forms on the manifold $M(\mathrm{GD}(\mathrm{sl}(n,\mathbb{R}))) \cdot Q^*$ valued in the line bundle $\tilde{E}_{h,c}$ generated by W-antighosts; *semi-infinite W-differential Banks–Peskin forms* are defined with respect to the $\mathbb{Z}$-grading on the Lie quasi(pseudo)algebra $\mathcal{GM}^{\mathbb{C}}_{n;(\mathrm{enl})}$.

The space of W-differential Banks–Peskin forms will be denoted by

$$\Omega_{\mathrm{BP}}(M(\mathrm{GD}(\mathrm{sl}(n,\mathbb{R}))) \cdot Q^*; \tilde{E}_{h,c})$$

and the space of semi-infinite W-differential Banks–Peskin forms by

$$\Omega^{\mathrm{SI}}_{\mathrm{BP}}(M(\mathrm{GD}(\mathrm{sl}(n,\mathbb{R}))) \cdot Q^*; \tilde{E}_{h,c})\,.$$

Remark 1. The Lie quasi(pseudo)algebra $\widehat{\mathcal{GM}}^{\mathbb{C}}_{n;(\mathrm{enl})}$ acts on the space

$$\Omega^{\mathrm{SI}}_{\mathrm{BP}}(M(\mathrm{GD}(\mathrm{sl}(n,\mathbb{R})))\cdot Q^*;\tilde{E}_{h,c})$$

with the central charge $c - 2(2n^3 - n - 1)$.

Definition 2.

1. *Siegel W-string fields* are elements of the space dual to the space

   $$\Omega^{\mathrm{SI}}_{\mathrm{BP}}(M(\mathrm{GD}(\mathrm{sl}(n,\mathbb{R})))\cdot Q^*;\tilde{E}_{h,c})$$

   of semi-infinite W-differential Banks–Peskin forms.
2. The W-BRST-*operator* is the operator Q_{BRST} in the space of Siegel W-string fields dual to the exterior covariant derivative D in the space of semi-infinite W-differential Banks–Peskin forms.

Remark 2. $Q^2_{\mathrm{BRST}} = 0$ iff $c = 2(2n^3 - n - 1)$.

Let us now study the aspects related to the gauge-invariance of W-string fields in the sense of Ref. [1] (see also Ref. [31]).

Namely, the space $\Omega^{\mathrm{SI}}_{\mathrm{BP}}(M(\mathrm{GD}(\mathrm{sl}(n,\mathbb{R})))\cdot Q^*;\tilde{E}_{h,c})$ of semi-infinite W-differential Banks–Peskin forms may be considered a space of holomorphic sections of a certain bundle over $M(\mathrm{GD}(\mathrm{sl}(n,\mathbb{R})))$, which will be called *Fock-plus-ghost bundle* and denoted by $\mathrm{FG}_{h,c}(M(\mathrm{GD}(\mathrm{sl}(n,\mathbb{R}))))$. Fibers of the vector bundle $\mathrm{FG}_{h,c}(M(\mathrm{GD}(\mathrm{sl}(n,\mathbb{R}))))$ over points x of the flag manifold $M(\mathrm{GD}(\mathrm{sl}(n,\mathbb{R})))$ for the Gelfand–Dickey algebra $\mathrm{GD}(\mathrm{sl}(n,\mathbb{R}))$ will be denoted by $\left(\mathrm{FG}_{h,c}\right)_x(M(\mathrm{GD}(\mathrm{sl}(n,\mathbb{R}))))$.

There exists a set $\{P_x\}$ of natural gauge-fixing projectors

$$P_x : \mathcal{O}(\mathrm{FG}_{h,c}(M(\mathrm{GD}(\mathrm{sl}(n,\mathbb{R}))))) \mapsto \left(\mathrm{FG}_{h,c}\right)_x(M(\mathrm{GD}(\mathrm{sl}(n,\mathbb{R}))))$$

(here the flag manifold $M(\mathrm{GD}(\mathrm{sl}(n,\mathbb{R})))$ is interpreted as a space of internal gauge degrees of freedom, cf. [31]) and a set $\{I_x\}$ of embedding operators

$$I_x : \left(\mathrm{FG}_{h,c}\right)_x(M(\mathrm{GD}(\mathrm{sl}(n,\mathbb{R})))) \mapsto \mathcal{O}(\mathrm{FG}_{h,c}(M(\mathrm{GD}(\mathrm{sl}(n,\mathbb{R}))))),$$

which satisfy the following two properties:

(1) $P_x I_x = \mathrm{id}$;

(2) $(\widehat{\mathcal{GM}}^{\mathbb{C}}_{n;(\mathrm{enl})})_+(x)I_x = 0$, where $(\widehat{\mathcal{GM}}^{\mathbb{C}}_{n;(\mathrm{enl})})_+(x)$ is the natural splitting of the exact sequence

$$0 \longrightarrow \left(\widehat{\mathcal{GM}}^{\mathbb{C}}_{n;(\mathrm{enl})}\right)_0(x) \longrightarrow \widehat{\mathcal{GM}}^{\mathbb{C}}_{n;(\mathrm{enl})}$$
$$\longrightarrow \widehat{\mathcal{GM}}^{\mathbb{C}}_{n;(\mathrm{enl})}/\left(\widehat{\mathcal{GM}}^{\mathbb{C}}_{n;(\mathrm{enl})}\right)_0(x) \longrightarrow 0,$$

where $(\widehat{\mathcal{GM}}^{\mathbb{C}}_{n;(\mathrm{enl})})_0(x) = \{v \in \widehat{\mathcal{GM}}^{\mathbb{C}}_{n;(\mathrm{enl})} : v(x) = 0\}$.

One may define the *W-string Gauss–Manin connection* in $\mathrm{FG}_{h,c}(M(\mathrm{GD}(\mathrm{sl}(n,\mathbb{R}))))$ as

$$\nabla_v\varphi(x) = \lim_{t\to\infty} t^{-1}(P_x I_{x+tv(x)} P_{x+tv(x)}\varphi - P_x\varphi),$$
$$v \in \mathrm{Vect}(M(\mathrm{GD}(\mathrm{sl}(n,\mathbb{R})))), x \in M(\mathrm{GD}(\mathrm{sl}(n,\mathbb{R}))).$$

This connection may be also defined by a means of *W-string Kostant–Blattner–Sternberg pairings*. Namely, the Fock space $F(M(\mathrm{GD}(\mathrm{sl}(n,\mathbb{R}))), \mathrm{FG}_{h,c})$ possesses a (pseudo)hermitean metric (cf. [13, Juriev; 31]). If such a metric is non-degenerate then it induces a metric $(\cdot,\cdot)$ on the space $\mathcal{O}(\mathrm{FG}_{h,c}(M(\mathrm{GD}(\mathrm{sl}(n,\mathbb{R})))))$. W-string Kostant–Blattner–Sternberg pairings are the mappings $B_{x,y}(\cdot,\cdot)$ from the tensor product of $(\mathrm{FG}_{h,c})_x(M(\mathrm{GD}(\mathrm{sl}(n,\mathbb{R}))))$ and $(\mathrm{FG}_{h,c})_y(M(\mathrm{GD}(\mathrm{sl}(n,\mathbb{R}))))$ to $\mathbb{C}$, such that $B_{x,y}(\varphi,\psi) = (I_x\varphi, I_y\psi)$.

The W-string Gauss–Manin connection ∇ may be expressed via W-string Kostant–Blattner–Sternberg pairings as follows:

$$\nabla_v\Phi(x) = 0 \ \text{ iff } B_{x+tv(x),x}(\Phi(x+tv(x)),\psi) = B_{x,x}(\Phi(x),\psi) + o(t)$$
$$\text{for all } \psi \in \left(\mathrm{FG}_{h,c}\right)_y (M(\mathrm{GD}(\mathrm{sl}(n,\mathbb{R})))),$$

where $\Phi(x)$ is a short notation for $P_x(\Phi)$.

Definition 3 (cf. [31]).
1. A covariantly constant section of the Fock-plus-ghost bundle

   $$\mathrm{FG}_{h,c}(M(\mathrm{GD}(\mathrm{sl}(n,\mathbb{R}))))$$

   over the flag manifold the $M(\mathrm{GD}(\mathrm{sl}(n,\mathbb{R})))$ for Gelfand–Dickey algebra $\mathrm{GD}(\mathrm{sl}(n,\mathbb{R}))$ is called *Bowick–Rajeev W-vacuum*.
2. The space dual to the space of Bowick–Rajeev vacua is called *the space of gauge-invariant Siegel W-string fields*.

Unfortunately, Bowick–Rajeev W-vacua (or, equivalently, gauge-invariant Siegel W-string fields) do not exist. *The phenomenon of geometric non-cancellation of the Bowick–Rajeev W-anomaly may be considered an explanation of the fact that Gelfand–Dickey brackets cannot be quantized only by addition of a central term.* The global structural change of commutation relations is necessary. Let us now describe the process of *operator cancellation* of the Bowick–Rajeev anomaly, which transforms Gelfand–Dickey algebras or enlarged Gervais–Matsuo quasi(pseudo)algebras into W_N-algebras (it should be mentioned that the problem of quantization of Lie quasi(pseudo)algebras was discussed in another context in the book of M.V. Karasev and V.P. Maslov [33]). Of course, such operator cancellation does not provide us with a BRST-operator for W_N-algebras.

Let us consider an arbitrary point x of the flag manifold $M(\mathrm{GD}(\mathrm{sl}(n,\mathbb{R})))$ for the Gelfand–Dickey algebra $\mathrm{GD}(\mathrm{sl}(n,\mathbb{R}))$. Let us embed the fiber

$$(\mathrm{FG}_{h,c})_x(M(\mathrm{GD}(\mathrm{sl}(n,\mathbb{R}))))$$

of the Fock-plus-ghost bundle $\mathrm{FG}_{h,c}(M(\mathrm{GD}(\mathrm{sl}(n,\mathbb{R}))))$ over x into the space

$\mathcal{O}(\mathrm{FG}_{\boldsymbol{h},c}(M(\mathrm{GD}(\mathrm{sl}(n,\mathbb{R})))))$

by use of I_x. Now define the action of the real form $\widehat{\mathcal{GM}}_{n;(\mathrm{enl})}$ of the enlarged Gervais–Matsuo quasi(pseudo)algebra $\widehat{\mathcal{GM}}^{\mathbb{C}}_{n;(\mathrm{enl})}$ on V as follows:

$$v(\phi) = P_x \nabla_v I_x \phi, \quad \phi \in \left(\mathrm{FG}_{\boldsymbol{h},c}\right)_x (M(\mathrm{GD}(\mathrm{sl}(n,\mathbb{R})))), v \in \widehat{\mathcal{GM}}_{n;(\mathrm{enl})}.$$

In fact, it is not an action because the commutation relations in the real form of the Gervais–Matsuo quasi(pseudo)algebra are broken, moreover, the object we have obtained is no longer a Lie quasi(pseudo)algebra, but an ordinary algebra of operators.

It is just remarkable that the obtained algebras are (after a complexification) just W_N-algebras of Refs. [3].

3. Conclusions

Now, as soon as our plan is performed, the results may be summarized.

It appeared that the main objects of the infinite dimensional W-geometry of a second quantized free string are not infinite dimensional groups, Lie algebras and their homogeneous spaces as they used to be in Ref. [1] but infinite dimensional Lie quasi(pseudo)-algebras (various modifications of the Gervais–Matsuo quasi(pseudo)algebra $\mathcal{GM}_n^{\mathbb{C}}$ of classical (infinitesimal) W-transformations), nonlinear Poisson brackets and related geometrical structures.

As a consequence, there exists a geometrical non-cancellation of the Bowick–Rajeev anomaly (the absence of gauge-invariant Siegel W-string fields). Operator cancellation provides us with a transformation of classical Lie quasi(pseudo)algebras $\widehat{\mathcal{GM}}^{\mathbb{C}}_{n;(\mathrm{enl})}$ (central extended enlarged Gervais–Matsuo algebras) into quantum W_N-algebras.

It should be mentioned that, since realistic W-string field theory is essentially the theory of a self-interacting string field (see e.g. Ref. [41]), free W-string field theory may be considered only as a starting point of the perturbation approach to it. Consequently, the infinite dimensional W-geometry of a second quantized free string may be also regarded as a zero-approximation of the noncommutative geometry of a self-interacting W-string field.

Acknowledgement

The author thanks Prof.J.-L. Gervais and the collective of the Laboratoire de Physique Théorique de l'École Normale Supérieure (Paris) for a friendly imaginative, creative atmosphere and hospitality, without which the writing of this paper would have been impossible. The author also specially thanks Prof.J.-L. Gervais, whose ideas on differential W-geometry provoked him to realize the plan of the article. Also the author is indebted to all participants of the seminar on quantum field theory (especially, Prof. A.Yu. Morozov and A.O. Radul) at the Institute for Theoretical and Experimental Physics (Moscow), where the problems related to W-symmetries were discussed many times. The author thanks M.I. Golenishcheva-Kutuzova, B.A. Khesin and V.Yu. Ovsienko, who being participants of the seminar on representation theory at the Department of Mathematics of Moscow State University, introduced the author into the theme, Prof. M.V. Saveliev (Institute of High Energy Physics, Serpukhov) for some conversations on differential W-geometry at LPTENS, Prof. L.V. Sabinin and P.O. Mikheev (Dept. of Mathematics, Univ. Peoples' Friendship, Moscow) for discussions on Lie quasi(pseudo)algebras, Lie quasi(pseudo)groups and related topics.

References

[1] D. Juriev, Infinite dimensional geometry and quantum field theory of strings. I. Infinite dimensional geometry of a second quantized free string, Alg. Groups Geom. 11(2) (1994) 145–179.

[2] D. Juriev, Infinite dimensional geometry and quantum field theory of strings. II. Infinite dimensional noncommutative geometry of self-interacting string field, Russian J. Math. Phys., to appear.

[3] A.B. Zamolodchikov, Infinite additional symmetries in 2-dimensional conformal field theory, Theor. Math. Phys. 65 (1985) 1205–1213;
A.B. Zamolodchikov and V.A. Fateev, Conformal quantum field theory in two dimensions having $\mathbb{Z}_3$-symmetry, Nucl. Phys. B 280 (1987) 644–660;
S.L. Lukyanov and V.A. Fateev, Conformally invariant models in 2-dimensional quantum field theory with $\mathbb{Z}_N$ symmetry, Zh. Exp. Theor. Fiz. 94(3) (1988) 23-37;
S.L. Lukyanov, Quantization of the Gel'fand–Dickey bracket, Funkt. Anal. Appl. 22 (1988) 255–263;
J.-L. Gervais, Systematic approach to conformal theories, Nucl. Phys. B (Proc. Suppl.) 5B (1988) 119–136;
J.-L. Gervais and A. Bilal, Conformal theories with non-linearly-extended Virasoro symmetries and Lie algebra classification, in: *Infinite Dimensional Lie Algebras and Lie Groups*, ed. V. Kac (World Scientific, Singapore, 1989) pp. 483–526; Systematic approach to conformal field systems with extended Virasoro symmetries, Phys. Lett. B 206 (1988) 412–420; Systematic construction of conformal theories with higher-spin Virasoro symmetries, Nucl. Phys. B 318 (1989) 579–630.

[4] V.Yu. Ovsienko and O.D. Ovsienko, Lie derivatives of order n on the line. Tensor meaning of the Gel'fand–Dickey bracket, Adv. Soviet Math. 2 (1991) 221–231;
V.Yu. Ovsienko and B.A. Khesin, Symplectic leaves of the Gel'fand–Dickey brackets and homotopy classes of nondegenerate curves, Funkt. Anal. Appl. 24 (1990) 33–40.

[5] O.S. Kravchenko and B.A. Khesin, A central extension of the algebra of pseudodifferential symbols, Funkt. Anal. Appl. 25 (1991) 152–154.

[6] A.O. Radul, Lie algebras of differential operators, their central extensions and W-algebras, Funkt. Anal. Appl. 25 (1991) 33–49; Non-trivial central extensions of Lie algebras of differential operators in two and higher dimensions, Phys. Lett. B 265 (1991) 86–91.

[7] I. Vaysburd and A. Radul, Differential operators and W-algebra, Phys. Lett. B 274 (1992) 317–322.

[8] I. Bakas, B. Khesin and E. Kiritsis, The logarithm of the derivative operator and higher spin algebras of W_∞ type, Commun. Math. Phys. 151 (1993) 233–243.

[9] V. Kac and A. Radul, Quasiaffine highest weight modules over the Lie algebra of differential operators on the circle, Commun. Math. Phys. 157 (1993) 429–458.

[10] B. Khesin and I. Zakharevich, Poisson–Lie group of pseudodifferential operators and fractional KP-KdV hierarchies, C.R. Acad. Sci. Paris I, 316 (1993) 621–626; Poisson–Lie group of pseudodifferential symbols; hep-th/9312088; Commun. Math. Phys., to appear.

[11] J.-L. Gervais and Y. Matsuo, W-geometries, Phys. Lett. B 274 (1992) 309–316; Classical A_n-W-geometry, Commun. Math. Phys. 152 (1993) 317–368;
J.-L. Gervais, W-geometry from chiral embeddings, J. Geom. Phys. 11 (1993) 293–304; Introduction to differential W-geometry, preprint LPTENS-93/38.

[12] P. Griffiths and S. Harris, *Principles of Algebraic Geometry* (Wiley Interscience, New York, 1978).

[13] A.A. Kirillov, A Kähler structure on K-orbits of the group of diffeomorphisms of a circle, Funkt. Anal. Appl. 21 (1987) 122–125;
A.A. Kirillov and D.V. Juriev, The Kähler geometry on the infinite dimensional homogeneous manifold $M = \mathrm{Diff}_+(\mathbb{S}^1)/\mathrm{Rot}(\mathbb{S}^1)$, Funkt. Anal. Appl. 20 (1986) 322–324; The Kähler geometry on the infinite dimensional homogeneous space $M = \mathrm{Diff}_+(\mathbb{S}^1)/\mathrm{Rot}(\mathbb{S}^1)$, Funkt. Anal. Appl. 21 (1987) 284–293; Representations of the Virasoro algebra by the orbit method, J. Geom. Phys. 5 (1988) 351–364 [reprinted in: *Geometry and Physics. Essays in honour of I.M. Gelfand.*, eds. S.G. Gindikin and S.G. Singer (Pitagora Editrice, Bologna and Elsevier Science, Amsterdam, 1991)];
D. Juriev, The vocabulary of geometry and harmonic analysis on the infinite-dimensional manifold $\mathrm{Diff}_+(\mathbb{S}^1)/\mathbb{S}^1$, Adv. Soviet Math. 2 (1991) 233–247; A model of Verma modules over the Virasoro algebra, Algebra Anal. 2(2) (1990) 209–226; An infinite dimensional geometry of the universal deformation of a complex disc, Russian J. Math. Phys. 2(1) (1994).

[14] D. Juriev, Quantum conformal field theory as infinite dimensional noncommutative geometry, Russian Math. Surveys 46(4) (1991) 135–163.

[15] A.Yu. Morozov, Strings – what are they?, Soviet Phys. Uspekhi 35 (1992) 671–714; Integrability and matrix models, Soviet Phys. Uspekhi 37 (1994) 1–109.

[16] D. Juriev, Quantum projective field theory: quantum field analogs of Euler formulas, Teor. Mat. Fiz. 92(1) (1992) 172–176; Quantum projective field theory: quantum field analogs of Euler–Arnold equations in projective G-hypermultiplets, Teor. Mat. Fiz. 98(2) (1994) 220–240.

[17] V.I. Arnold, *Mathematical Methods of Classical Mechanics* (Springer, 1976).

[18] E. Floratos and J. Iliopoulos, A note on the classical symmetries of the closed bosonic membranes, Phys. Lett. B 201 (1988) 237–240;
I. Antoniadis, P. Ditsas, E. Floratos and J. Iliopoulos, New realizations of the Virasoro algebra as membrane symmetries, Nucl. Phys. B 300 (1988) 549–558.

[19] I.M. Gelfand and D.B. Fairlie, The algebra of Weyl symmetrized polynomials and its quantum extension, Commun. Math. Phys. 136 (1991) 487–499.

[20] D.B. Fairlie, P. Fletcher and C.K. Zachos, Trigonometric structure constants for new infinite-dimensional algebras, Phys. Lett. B 218 (1989) 203–206;
D.B. Fairlie and C.K. Zachos, Infinite dimensional algebras, sine brackets and $\mathrm{SU}(\infty)$, Phys.Lett. B 224 (1989) 101–107;
D.B. Fairlie, C.K. Zachos and P. Fletcher, Infinite dimensional algebras and a trigonometric basis for the classical Lie algebras, J. Math. Phys. 31 (1990) 1088–1094.

[21] V. Kac and D.H. Peterson, Spin and wedge representations of infinite-dimensional Lie algebras and groups, Proc. Nat'l Acad. Sci. USA 78 (1981) 3308–3312.

[22] D.B. Fairlie and J. Nuyts, Deformations and renormalization of W_∞, Commun. Math. Phys. 134 (1990) 413–419.

[23] I. Bakas, The large-N limit of extended conformal symmetries, Phys. Lett. B 228 (1989) 57–63.

[24] C.N. Pope, X. Shen and L.J. Romans, The complete structure of W_∞, Phys. Lett. B 236 (1990) 173–178; A new higher-spin algebra and the lone-star product, Phys. Lett. B 242 (1990) 401–406; W_∞ and the Racah–Wigner algebra, Nucl. Phys. B 339 (1990) 191–221.

[25] I. Bakas, The structure of the W_∞-algebra, Commun. Math. Phys. 134 (1990) 487–508.

[26] I.N. Bernstein, I.M. Gelfand and S.I. Gelfand, Models of representations of Lie groups, Trudy Semin. I.G. Petrovskogo 2 (1976) 3–21.

[27] B.L. Feigin, The Lie algebras $gl(\lambda)$ and the cohomology of the Lie algebra of differential operators, Russian Math. Surveys 35(2) (1988) 169–170.

[28] D.V. Juriev, Complex projective geometry and quantum projective field theory, Teor. Mat. Fiz. 101(3) (1994), to appear.

[29] D. Juriev, Infinite dimensional geometry and closed string-field theory, Lett. Math. Phys. 22 (1991) 1–6.

[30] D. Juriev, Cohomology of the Virasoro algebra with coefficients in string fields, Lett. Math. Phys. 19 (1990) 355–356.

[31] D. Juriev, Gauge invariance of Banks–Peskin differential forms (flat background), Lett. Math. Phys. 19 (1990) 59–64.

[32] I. Batalin, Quasigroup construction and first class constraints, J. Math. Phys. 22 (1981) 1837–1850.

[33] M.V. Karasev and V.P. Maslov, *Nonlinear Poisson Brackets. Geometry and Quantization* (Nauka, Moscow, 1991).

[34] L.V. Sabinin, On the equivalence of the category of loops and the category of homogeneous spaces, Soviet Math. 13 (1972) 533–536.
[35] P.O. Mikheev and L.V. Sabinin, On the infinitesimal theory of local analytic loops, Soviet Math. 36 (1988) 545–548.
[36] J.-L. Gervais and M.V. Saveliev, *W*-geometry of the Toda systems associated with non-exceptional simple Lie algebras, preprint LPTENS-93/47.
[37] P.O. Mikheev and L.V. Sabinin, Smooth quasigroups and geometry, VINITI Problems in Geometry 20 (1988) 75–110 [and several unpublished texts on "nonlinear geometric algebra"].
[38] Yu.I. Manin, *Cubic Forms: Algebra, Geometry, Arithmetics* (North-Holland, Amsterdam, 1974).
[39] E. Cartan, Sur les domaines bornés homogènes de l'espace de n variables complexes, Abh. Math. Seminar Hamburg 11 (1935) 116–162 [reprinted in: *Oeuvres Complètes* (Gauthier-Villars, Paris, 1953)].
[40] M.J. Bowick and S.G. Rajeev, String theory as the Kähler geometry of loop space, Phys. Rev. Lett. 58 (1987) 535–538; The holomorphic geometry of closed bosonic string theory and $\mathrm{Diff}(\mathbb{S}^1)/\mathbb{S}^1$, Nucl. Phys. B 293 (1987) 348–384.
[41] A. Bilal and J.-L. Gervais, Non-linearly extended Virasoro algebras: new prospects for building string theories, Nucl. Phys. B 326 (1989) 222–236.